The Multiple-Volunteers Principle for Unpleasant Tasks and for Pleasant Tasks *

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Abstract

We present a class of simple transfer-free rules that are very effective tools for assigning an unpleasant task among a group of agents: every agent decides whether or not to "volunteer"; if the number of volunteers exceeds a threshold number, the task is assigned to a volunteer; if the number is below the threshold, the task is assigned to a non-volunteer.

In a setting in which agents have non-trivial preferences over *who* performs the task, such a threshold rule is utilitarian optimal across all binary-action rules. In a large group, the first best is reached approximately via a threshold rule with a large threshold. Threshold rules have a robust-improvement property: any rule with a non-extreme threshold always has an equilibrium that yields a strict interim Pareto improvement over a random task assignment. We show that assigning the task to a non-volunteer rather than randomly among all agents if the threshold is not reached is crucial for this result. Such a uniformly-random default, however, is utilitarian optimal if ex-post participation constraints are imposed, and is still good enough to approximate the first best in a large population. The results can be adapted to the problem of assigning a pleasant task.

1 Introduction

Imagine a group of people from which one must be selected as the performer of a task. Examples include the selection of a person to stand first in line in a dangerous (e.g., military battle)

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situation, the selection of the salesperson who deals with the unpleasant customer, or the selection of the chairperson of a department at a university. The performance of the task is similar to the provision of a public good, with the two special features that this particular public good *must* be provided without delay, and that no monetary transfers are feasible.

The optimal task assignment depends on the agents' preferences, which are private information. If each person was solely interested in minimizing the probability of being personally selected for the task, then, due to incentive constraints and given the lack of remuneration, no screening of any private information would be possible, whatever the prevailing rule is. Fortunately, in many situations people are also genuinely interested in *who* performs the task. This can be because some people would perform the task at a higher quality than others, or because people are at least slightly altruistic and care about others' costs of being selected. Formally, we assume that a general affine function of the performing agent's (privately known) payoff determines the other agents' payoffs. Both the standard private-values and complete-information model are limits of our setup. Perhaps surprisingly, theoretical analyses of our setup are rare (see the literature discussion below).

What are practical task-assignment rules in such situations? In many problems of mechanism design, ex-post incentive-compatible (EPIC) rules are attractive because of their robustness with respect to beliefs. In our setting, however, the only anonymous EPIC rule is the uniformly-random task assignment.[1] We take a different route and restrict attention to rules that are practical due to their small number of available choices. Specifically, we focus on rules in which agents choose simultaneously from just *two* available actions, using symmetric Bayes-Nash equilibrium as a solution concept. Bayes-Nash equilibrium is particularly plausible for binary-action rules because with just two alternatives a player can quickly find a best response via trial and error. We show that these rules can be quite powerful.[2]

Our results, which are derived under the assumption of continuously distributed private information, are as follows. First, we identify a one-parameter class of rules called threshold rules which always contains a rule that is utilitarian-optimal among the binary-action rules. Second, we show that any non-extreme threshold rule always allows for a strict interim Pareto-improvement over a uniformly-random task assignment, even if the task-performance cost is very high, or if the agents are just a tiny bit altruistic; we call this the *robust-improvement* property. Third, we consider the limit case of a large population. Any given threshold number leads to an efficiency loss in the limit, but with a sequence of increasingly large thresholds the first best can be approximated arbitrarily closely. Proving our results requires novel methods; the equilibrium conditions for binary-action games, though one-dimensional, are non-linear, and we have to compare the equilibrium welfare across all such games.

[1]Feng, Niemeyer, and Wu (2022) derive a similar conclusion in a class of allocation problems without monetary transfers, showing that only constant rules are EPIC. Their technical assumptions, however, exclude our setting. We use a different proof technique; see the Appendix.

[2]Implementing a binary-action rule can involve the commitment to implement a task assignment that is inefficient, given the information revealed by the agents' actions. Our assumption that the task must be assigned without delay helps justify this commitment, as the possibility of renegotiation usually relies on the option to delay (cf. Coase (1972) in the context of monopoly).

The binary-action rule that first comes to mind as a potential improvement over a uniformly-random assignment of the task is the *any-volunteer rule*: all agents are asked simultaneously about who would like to "volunteer"; if at least one agent volunteers, the task is assigned randomly among the volunteering agents; if no agent volunteers, the task is assigned randomly among all agents. The any-volunteer rule often leads to expected (utilitarian) welfare gains relative to a uniformly-random assignment because agents can self-select into volunteers and non-volunteers according to their private information. But, and this is the starting point of our paper, other equally simple rules can yield even higher welfare.

For example, a *two*-volunteers rule may be used which stipulates that the task is assigned among the non-volunteers unless at least two volunteers come forward. As we show, if performing the task is sufficiently costly, or if agents' altruism is small, then in the unique equilibrium of the any-volunteer rule no volunteers will come forward so that a uniformly-random task assignment obtains, but the two-volunteers rule always has an equilibrium in which all types of all agents (volunteers as well as non-volunteers) are strictly better off than with a uniformly-random assignment; this holds, in fact, for any dynamically stable equilibrium. This robust-improvement property generalizes to threshold rules with thresholds higher than two. The basic intuition is that by committing to select a non-volunteering agent for the task if the threshold is not reached, the designer effectively creates a cost of non-volunteering that always keeps up some volunteering activity. The principle of requiring multiple volunteers although one agent would be sufficient to perform a task may be called the multiple-volunteers principle.

It is instructive to compare any threshold rule to its *uniformly-random-default* counterpart, in which the task is assigned randomly among all agents—rather than only the non-volunteers—if the threshold is not reached. These rules lack the "cost of non-volunteering". We show that the threshold rules with uniformly-random default do not share the robust-improvement property: like in the any-volunteer rule, if performing the task is rather costly or if agents' altruism is rather small, the unique equilibrium is such that no volunteers will come forward. On the other hand, many threshold rules with uniformly-random default have a desirable equilibrium property that is not shared by the standard threshold rules: even after learning how many volunteers have come forward, each agent type still expects to receive at least the uniformly-random assignment payoff. Augmenting the utilitarian designer's problem with corresponding ex-post participation constraints, a threshold rule with uniformly-random default becomes utilitarian optimal among binary-action rules. Our large-population welfare results, however, hold for both classes of threshold rules, with and without uniformly random default.

Literature

Volunteering, according to Wilson (2000), is "any activity in which time is given freely to benefit another person, group or cause." Implicit here is the assumption that there is little or no remuneration for the activity. Volunteering plays an important role in many different areas of any modern economy.[3] While many volunteering activities are informal, in the UK the economic

[3]To give just one example, the German association of hospices reports that most of the 120.000 individuals working under their roof are volunteers who are not remunerated, see https://www.dhpv.de/themen_hospiz-

value of formal volunteering alone is estimated at £39 billion, according to the report by Low, Butt, Ellis, and Davis Smith (2007). An extensive sociological literature focuses on the nature of and motivations for volunteering (see the survey by Wilson (2000)).

Volunteering as a strategic game has mainly been studied in private-value settings or under complete information (both of which are limits of our model). In the simplest version of the volunteers' game, the task is a public good that is provided if and only if at least one volunteer comes forward, and every volunteer pays the (homogeneous) cost of providing the public good. Olson (2009, first edition: 1965) conjectured that the equilibrium probability of a volunteer coming forward is decreasing in the group size. Olson's conjecture was proven by Diekmann (1985) assuming complete information. Nöldeke and Peña (2020) generalize this result for volunteers' games where multiple contributors may be required to produce the public good.[4] In an incomplete-information setting with private values, Bergstrom and Leo (2015) characterize the type distributions for which Olson's conjecture holds. We obtain, in essence, a conclusion opposite to the Olsen conjecture: volunteering can be organized more efficiently in a large population than in a small population. Formally, in our second best, at any finite population size the probability that the task is provided at a quality that is ϵ-close to the highest among all agents is strictly below 1 if $\epsilon > 0$ is sufficiently close to 0, but this probability tends to 1 in the large-population limit.

The volunteers game has also been studied in a *coordinated* version meaning that one of the volunteers is randomly selected to provide the public good (this rule is similar to the any-volunteers rule with the crucial difference that the public good is not provided if no volunteer comes forward). Weesie and Franzen (1998) prove Olson's conjecture for this game, assuming complete information.[5]

In the private-values or complete-information settings discussed above, incentives for volunteering arise from the threat that the public good is not provided *at all* rather than, as in our setting, from the threat that the public good is provided by the *wrong agent*. Our model could be extended to allow for the possibility that the public good is not provided at all, but given our model of preferences and private information, the wrong-agent threat is already a powerful incentive device.

Our preference model in which a general affine function of the performing agent's type determines the other agents' payoffs first appears in Li, Rantakari, and Yang (2016). They restrict

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[4]De Jaegher (2020) shows that there are cases in which (i) multiple, say k, contributors are needed to produce the public good, (ii) in equilibrium no contributor will come forward, and (iii) a positive equilibrium level of contributing can be restored by committing to produce the public good only if at least $k+1$ contributors come forward. This result can be seen as an instance of the multiple-volunteers principle.

[5]Makris (2009), still maintaining complete information, characterizes the symmetric equilibria in the coordinated volunteers game where multiple contributors may be required to produce the public good. Requiring multiple contributors implies that there exists an additional, unattractive, equilibrium in which nobody contributes. Makris shows that this equilibrium vanishes if two conditions are jointly satisfied: a small expected fraction of the population are "warm glow" givers for whom contributing is a dominant action, and the population size is Poisson distributed; precisely, he shows that there always exists a unique symmetric equilibrium, and in this equilibrium every regular (i.e., not warm-glow) agent contributes with a positive probability if the expected fraction of warm-glow givers is sufficiently small.

attention to settings with two agents, but—in contrast to us—allow preferences to be ex-ante asymmetric across agents. The focus is on cheap talk rather than on mechanism design, where the receiver's preferences correspond to our planner's objective. It is shown that, in the receiver-preferred equilibrium, the precision of both agents' communication is increasing in each agent's preference alignment with the planner.

Bhaskar and Sadler (2019) consider transfer-free mechanism design (without a binary-action restriction) among agents with private information and multiplicative, but not additive, preference misalignment, which is a special case of the pleasant-task variation of our preference model. They consider a public good that may or may not be provided. They identify an interesting class of mechanisms called *bin-mechanisms*. Simple bin mechanisms with two bins are equivalent to our threshold rules. Bhaskar and Sadler (2019) also define non-simple ("better") bin mechanisms and show that one such mechanism is welfare maximizing among all transfer-free rules if the distribution of types is uniform.

The altruism-case of our model can be seen as an example of private provision of a public good with other-regarding preferences, where each agent is privately informed about her provision cost. We did not find a Bayesian-Nash analysis of this particular setting in the literature. Palfrey and Rosenthal (1988) analyze volunteering assuming that each agent has a private "warm-glow" benefit from contributing. Kucuksenel (2012) considers mechanism design with transfers when agents are altruistic in the sense of caring about a weighted average of their own and others' material payoffs. He characterizes interim-efficient mechanisms. In his public-good application, he assumes that each agent is privately informed about her willingness to pay and provides comparative statics with respect to the degree of altruism.

A threshold rule that requires multiple volunteers can be seen as a new design that provides an alternative to the common any-volunteer rule. This is in the spirit of many other examples of mechanism design, where new designs have been proposed (and often implemented in practise) as replacements of existing ones. Alternatively, a multiple-volunteers rule can be an implicit social norm that exists without a social planner. To quote an example from biology, Magurran and Pitcher (1987) report that in fish swarms threatened by a predator volunteers are needed to "inspect" whether the predation threat is real. Some individual fish volunteer as inspectors to share a risk that would be borne by the entire swarm if the inspection did not happen (cf. Figure 8 in Magurran and Pitcher (1987)). To the extent that a volunteering group consists of multiple animals when one is technically sufficient to inspect the predator, this can be interpreted as an instance of a multiple-volunteers rule with uniformly random default.

Closely related to the multiple-volunteers principle as a design element is the idea of a conditional volunteering strategy by an individual agent. Schelling (2006) casts the idea of "volunteering if 20 others do likewise" (p. 95). Reischmann and Oechssler (2018) and Oechssler, Reischmann, and Sofianos (2022) provide a theoretical and experimental analysis of mechanisms that explicitly allow for conditional volunteering in the context of the provision of a continuous public good. Agents can send messages of the form "I am willing to contribute x units to the public good if in total y units are contributed". If we re-interpret x and y as probabilities of a contribution, then volunteering in our model is in effect such a conditional-contribution message, where x and y are determined by the equilibrium conditions. In the approach of Reischmann and Oechssler, however, agents are free to choose x and y. Considering settings with complete information or private values, they show that conditional-contribution mechanisms are powerful tools for improving welfare.[6]

An important strand of literature that focusses on binary-action rules in settings with (often) continuously distributed types is the literature on voting over two alternatives, where each agent can only vote for either alternative (abstention may constitute a third action). This literature presents many positive results for large-population limits, concerning either full information aggregation (e.g., Feddersen and Pesendorfer (1997), Krishna and Morgan (2011)) or first-best utilitarian welfare (e.g., Ledyard and Palfrey (2002), Krishna and Morgan (2015)). These results are different from our limit result. Voting limit results typically rely on the law of large numbers, while our result relies on bounds for tail probabilities of the Poisson distributions. This is because in the voting models, the planner's (or pivotal voter's) optimal choice in a large population relies on some *average* of the individuals' private information, while in our case the identity of the agent with the *maximum* type is of interest.

In between our assumption that the public good must be provided now and the opposite assumption that it can be avoided lies the possibility that the provision can be delayed, leading to discounted costs and benefits. The possibility of delay naturally leads to a war-of-attrition game in which each agent waits, or engages in some other costly search process, until someone agrees to provide the service. In a private-values setting with heterogenous costs, such a game has been analyzed by Bliss and Nalebuff (1984).[7] In equilibrium, it is often the "right" person who volunteers first, e.g., the one who has the lowest cost of providing the service, but it can also be the one who has the highest cost of waiting, and substantial waiting costs may have to be incurred before a volunteer is found.

2 Model

A task of public interest needs to be allocated among a group of agents $1, \ldots, n$, where $n \geq 2$.

2.1 Preferences and information

Each agent is privately informed about her *type*. Each agent's type is independently distributed on an interval $[t_L, t_H] \subseteq \mathbb{R}$ according to some strictly increasing and continuous cumulative distribution function F. We denote the expected type by

$$\bar{t} = \int_{t_L}^{t_H} t \mathrm{d}F(t).$$

[6]Earlier contributions have proposed provision-point mechanisms as simple rules that may lead to increased public good provision in the private-values setting. In these mechanisms, the cost of the public good acts as a threshold, as the public good is provided only if the sum of contributions equals or exceeds its cost (Bagnoli and Lipman, 1989).

[7]See also Bilodeau and Slivinski (1996) for a related model with complete information. See Klemperer and Bulow (1999) for a general approach to war-or-attrition games, and see LaCasse, Ponsati, and Barham (2002) and Sahuguet (2006) for more special extensions.

If agent i performs the task, then her utility is t_i and each other agent's utility is $\alpha t_i + \beta$. Agents are expected-utility maximizers.

There are several special cases of this model that illustrate possible interpretations. One is that the cost of providing the task is the same for all agents, but each agent is privately informed about her *ability* at performing the task. If the task is performed by an agent of ability θ, then every agent obtains the benefit θ. In addition, the performing agent bears a cost c. If each agent's ability type is drawn from an interval $[\theta_L, \theta_H]$, then $[t_L, t_H] = [\theta_L - c, \theta_H - c]$. This case fits our general model with $\alpha = 1$ and $\beta = c$.[8] For tiny ability differences ($\theta_L \approx \theta_H$), the setting approximates one with complete information.

Another special case of our model is that the agents differ in the cost of providing the service, have identical abilities, and are at least somewhat altruistic in the sense that they feel bad if someone else has to incur a cost for their benefit. In this case, we may think of the ability or benefit as being normalized to 0 and the provision cost being $-t$, leading to parameters $\alpha \in (0, 1)$ and $\beta = 0$. For small values of α, agents care just a tiny bit about the cost of others; such a setting approximates one with private values.

The ability to provide the task can also be related to the effort cost that is needed for the task. For concreteness, suppose that the task can be successful or not, where $e \in [0, 1]$ denotes the probability of success and is an unobservable effort choice by the selected agent. In case of a success each agent receives a reward payoff of $R \in (0, 1)$. Agents differ in their cost of effort, which is $c(e) = \frac{e^2}{\theta}$, with $\theta \in [1, 2]$. An agent of type θ who chooses effort e receives $eR - c(e)$ and provides an expected utility eR to the others. Effort choice of the selected agent will be $e(\theta) = \theta R/2$ such that utility from being selected is $\frac{R^2\theta}{4}$, while the benefit for the others is $\frac{R^2\theta}{2}$. Hence, with $t = \frac{R^2\theta}{4}$, $\alpha = 2$ and $\beta = 0$ this example with effort choice fits our framework.

In order to keep the presentation concise and to reduce the number of cases to distinguish, we assume that the performer's utility is congruent with the other agents' utilities, and that agents of all types prefer someone of their own type providing the service over doing it themselves. The latter condition means that there is a free-riding problem and finding a volunteer for the task is indeed difficult or, put differently, that the task is unpleasant.

Assumption 1.
$$\alpha > 0 \quad \textit{and} \quad t < \alpha t + \beta \textit{ for all } t \in [t_L, t_H]. \tag{1}$$

In the heterogenous-ability case, Assumption 1 means that the cost $c > 0$. In the altruism case, it means that only strictly negative types are possible, $t_H < 0$. Assumption 1 is also satisfied in the above effort-choice example.

2.2 Mechanisms

A social planner or the group of agents themselves can commit to the rules of a mechanism to allocate the task among them. We are interested in symmetric equilibria of (anonymous)

[8]The heterogeneous-ability case can be generalized by allowing an agent's cost to be a function of her ability, that is, $c(\theta) = \gamma\theta + \gamma'$, where $\gamma < 1$ and γ' are commonly known parameters. This fits our model with $\alpha = 1/(1-\gamma)$ and $\beta = \gamma'/(1-\gamma)$.

mechanisms that allow only *two* messages. In a *binary mechanism*, each player chooses between two actions, denoted by "Y" and "N". We do not allow players to opt out because we will show that in any equilibrium each type of agent is at least as well off as in a *uniformly-random assignment*, by which we mean that the task is always assigned with equal probability to any agent, independently of the realized type profile.

Assuming anonymity, a binary mechanism is characterized by a list

$$p_1, \ldots, p_{n-1}.$$

For all $j = 1, \ldots, n-1$, the number p_j denotes the probability that the task is assigned to a randomly selected Y-player if there are j Y-players; with probability $1 - p_j$, it is assigned to a randomly selected N-player if there are j Y-players. If the number of Y-players is 0 or n, the task is assigned randomly among all agents, that is, each agent gets assigned the task with probability $1/n$.

The restriction to binary mechanisms is motivated by their simplicity. Each player can be imagined to find a best response action, e.g. by trial and error in an adaptive process the rest points of which are the equilibria. We will show that binary mechanisms, in spite of their simplicity, are quite powerful. In contrast, a restriction to ex-post incentive-compatible mechanisms with arbitrary action spaces, which would address similar concerns, only leaves the possibility of random assignment.

Lemma 1. *The only anonymous ex-post incentive-compatible (EPIC) task-assignment rule is the one that induces the uniformly-random assignment.*

The proof of this result can be found in the Appendix.

Given any mechanism $(p_1, \ldots, p_{n-1})$, a symmetric strategy profile is characterized by a function σ that determines the strategy for each agent, where $\sigma(t)$ denotes the probability that type $t \in [t_L, t_H]$ chooses Y. Given any symmetric strategy profile σ, the ex-ante probability that a given agent chooses Y is denoted by

$$y(\sigma) = \int \sigma(t) \mathrm{d}F(t).$$

The expected type of a Y-player in this equilibrium is denoted by

$$t_Y(\sigma) = \frac{1}{y(\sigma)} \int \sigma(t) t \mathrm{d}F(t) \quad \text{if } y(\sigma) > 0,$$

and of an N-player by

$$t_N(\sigma) = \frac{1}{1 - y(\sigma)} \int (1 - \sigma(t)) t \mathrm{d}F(t) \quad \text{if } y(\sigma) < 1.$$

Moreover, the expected benefit that accrues to every other agent if the task is assigned to an a-player is denoted by

$$u_a(\sigma) = \alpha t_a(\sigma) + \beta \quad \text{for } a = Y, N.$$

The expected utility $U_a(\sigma, t)$ of any type t taking action a, who anticipates that the other agents will use the strategy σ, is

$$U_Y(\sigma,t) = \sum_{j=0}^{n-1} B_{y(\sigma)}^{n-1}(j) \left(p_{j+1} \frac{j u_Y(\sigma) + t}{j+1} + (1 - p_{j+1}) u_N(\sigma) \right), \tag{2}$$

$$U_N(\sigma,t) = \sum_{j=0}^{n-1} B_{y(\sigma)}^{n-1}(j) \left(p_j u_Y(\sigma) + (1 - p_j) \frac{(n-j-1) u_N(\sigma) + t}{n-j} \right). \tag{3}$$

Using the binomial distribution, $B_y^{n-1}(j) = \binom{n-1}{j}(1-y)^{n-1-j} y^j$ denotes[9] the probability that, from the point of view of the given agent, j of the $n-1$ other agents choose Y, given that each of them chooses Y with probability y. We also use the notation $p_n = 1$ and $p_0 = 0$.

The function σ is an *equilibrium* if the following implications hold for all t:

$$\text{if } \sigma(t) > 0 \text{ then } U_Y(\sigma,t) - U_N(\sigma,t) \geq 0,$$
$$\text{if } \sigma(t) < 1 \text{ then } U_Y(\sigma,t) - U_N(\sigma,t) \leq 0.$$

Mechanism-equilibrium combinations $(p_1, \ldots, p_{n-1}, \sigma)$ and $(p'_1, \ldots, p'_{n-1}, \sigma')$ are *equivalent* if each type obtains the same expected utility in both combinations.

In the *uniformly-random assignment rule* given by $p_j = j/n$ for all j, any strategy is an equilibrium that induces the uniformly random assignment.

2.3 Welfare

The social planner's problem is to find a mechanism-equilibrium combination $(p_1, \ldots, p_{n-1}, \sigma)$ that maximizes utilitarian welfare,[10]

$$\sum_{j=0}^{n} B_{y(\sigma)}^{n}(j) \Big(p_j \big(t_Y(\sigma) + (n-1) u_Y(\sigma) \big) + (1-p_j) \big(t_N(\sigma) + (n-1) u_N(\sigma) \big) \Big).$$

Because the task cannot be avoided, the terms involving β add up to a constant additive term $(n-1)\beta$ in the utilitarian welfare. Dropping this constant, the planner maximizes

$$(1 + \alpha(n-1))\, W,$$

where we use the shortcut

$$W = \sum_{j=0}^{n} B_{y(\sigma)}^{n}(j) \left(p_j t_Y(\sigma) + (1-p_j) t_N(\sigma) \right).$$

[9] We use the convention that $0^0 = 1$.

[10] Given our focus on symmetric equilibria, this objective is equivalent to maximizing any agent's ex-ante expected utility.

for the expected type of the selected agent. Since $\alpha > 0$, the planner's objective boils down to maximizing W. In terms of our two leading interpretations, this means that the planner maximizes the expected ability or minimizes the expected cost of the selected agent.

3 Preliminary Results

In this section, we introduce a particular class of equilibria called partition equilibria that will play a central role. We formulate the planner's first- and second-best problems under the restriction to binary-action rules, and we show that, concerning the first best, the binary-action assumption vanishes to be restrictive in a large population.

3.1 Notation: selection-probability functions and expected utilities

We first introduce four auxiliary functions, h_Y, h_N, q_Y and q_N and discuss their basic properties. We will express the agents' expected utilities in terms of these functions. This has the advantage that binomial sums can remain hidden behind these functions for a large part of the analysis. Taking the point of view of a given agent who has chosen an action (Y or N), the functions q_Y and q_N describe the probability of personally getting assigned the task, and the functions h_Y and h_N describe the probability that someone in the group of Y-playing agents gets assigned the task.

Let $y \in [0,1]$ denote every other (i.e., not the given) agent's probability of playing Y. The probability that anyone of the Y-playing agents is selected, conditional on the event that the given agent plays Y, is denoted

$$h_Y(y) = \sum_{j=0}^{n-1} B_y^{n-1}(j) p_{j+1}. \tag{4}$$

Similarly, the probability that anyone of the Y-playing agents is selected, conditional on the event that the given agent plays N, is denoted by

$$h_N(y) = \sum_{j=0}^{n-1} B_y^{n-1}(j) p_j. \tag{5}$$

The probability that the given agent is selected herself if she chooses action $a = Y, N$ is denoted $q_a(y)$; i.e.,

$$q_Y(y) = \sum_{j=0}^{n-1} B_y^{n-1}(j) \frac{p_{j+1}}{j+1} \tag{6}$$

and

$$q_N(y) = \sum_{j=0}^{n-1} B_y^{n-1}(j) \frac{1-p_j}{n-j}. \tag{7}$$

As an example, consider the uniformly-random-assignment rule. Here, a computation that applies standard properties of the binomial distribution to the definitions (4)–(7) shows that[11]

$$\begin{aligned} h_Y(y) &= \frac{1 + y(n-1)}{n}, \quad h_N(y) = \frac{y(n-1)}{n}, \\ q_Y(y) &= \frac{1}{n}, \quad q_N(y) = \frac{1}{n}. \end{aligned} \tag{8}$$

Next we establish several useful algebraic relations between the auxiliary functions. These relations hold independently of the underlying mechanism. A particularly simple formula is available for expressing q_Y and q_N in terms of h_Y and h_N. To see it, suppose that all agents play Y with probability y. Then the probability that the task is assigned to a Y-player can be expressed in the form $yh_Y + (1-y)h_N$. Alternatively, the same probability can be expressed in the form nyq_Y because every Y-playing agent is selected with the same probability. Thus,

$$nyq_Y(y) = yh_Y(y) + (1-y)h_N(y). \tag{9}$$

Similarly, the probability that an N-player is selected is given by

$$n(1-y)q_N(y) = y(1 - h_Y(y)) + (1-y)(1 - h_N(y)). \tag{10}$$

Adding up the equations (9) and (10) confirms the ex-ante probability that any given agent is selected:

$$yq_Y(y) + (1-y)q_N(y) = \frac{1}{n}. \tag{11}$$

The derivatives of q_Y and q_N can again be related to the selection-probability functions.

Lemma 2. *Consider any mechanism and any* $0 < y < 1$. *Then*

$$q'_Y(y) = \frac{h_Y(y) - h_N(y) - q_Y(y)}{y}, \tag{12}$$

$$q'_N(y) = \frac{h_N(y) - h_Y(y) + q_N(y)}{1-y}. \tag{13}$$

The proof, which relies on standard properties of Bernstein polynomials, is relegated to the Appendix. Towards computing equilibria, it is crucial to evaluate an agent's expected-utility gain from playing Y versus playing N, assuming that all other agents use a strategy σ. The agent's expected utility can be written as

$$U_Y(\sigma, t) = (h_Y - q_Y)u_Y + q_Y \cdot t + (1 - h_Y)u_N \tag{14}$$

and

$$U_N(\sigma, t) = (1 - h_N - q_N)u_N + q_N \cdot t + h_N u_Y. \tag{15}$$

The functions on the right-hand sides are evaluated at $y(\sigma)$, which is omitted as an argument

[11] To understand the numerators in the formulas for h_Y and h_N, note that, from the point of view of a given agent, the expected number of other Y-players is equal to $y(n-1)$; by playing Y, the agent adds in herself (1+).

for the sake of brevity. The interpretation of these expressions is straightforward. Consider for instance the expected utility (14) from playing Y: the first term captures the payoff that arises from the event that the task is performed by a Y-player other than the agent herself, which happens with probability $h_Y - q_Y$; the second term captures the event that the agent is selected herself, which happens with probability q_Y, yielding the utility t; the third term captures the payoff that arises from the event that the task is performed by an N-player.

Combining the expressions (14) and (15) and cancelling terms, the utility gain from playing Y versus playing N is

$$U_Y(\sigma, t) - U_N(\sigma, t) = (q_Y - q_N)t + (h_Y - h_N - q_Y)u_Y - (h_Y - h_N - q_N)u_N. \qquad (16)$$

Note that the first term on the right-hand side is the only one that involves the agent's type t. The expression is linear in t, the sign being determined by $q_Y - q_N$. This gives us some intuition for the general shape of equilibria in this setting.

3.2 Partition equilibria

We now introduce a special class of equilibria, in which all agents with types below a cut-off type choose N and all agents with types above choose Y. A strategy σ is a *partition strategy* if there exists a cut-off type $\hat{t} \in [t_L, t_H]$ such that $\sigma(t) = 0$ for all $t < \hat{t}$ and $\sigma(t) = 1$ for all $t > \hat{t}$. A symmetric equilibrium in which the agents use a partition strategy is called a *partition equilibrium.* There are mechanisms for which no such partition equilibrium exists. If one exists, it is not clear whether it yields the highest welfare among all equilibria. We will show, however, that a social planner who is free to choose the mechanism can focus on partition equilibria without loss of generality (see Lemma 3).

Any partition strategy is characterized by the Y-playing probability $y = 1 - F(\hat{t})$. In the following, we will identify any partition strategy σ with the number $y = y(\sigma)$. Whenever we deal with a partition strategy $y \in [0, 1]$, we define the cut-off type by

$$\hat{t}(y) = F^{-1}(1 - y).$$

We adapt the notation for the expected type of a Y-player and N-player, respectively, and write

$$t_Y(y) = E[t|t \geq \hat{t}(y)] \text{ and } t_N(y) = E[t|t \leq \hat{t}(y)].$$

We define the continuous extensions $t_Y(0) = t_H$ and $t_N(1) = t_L$. The expected benefit created by an a-player is $u_a(y) = \alpha t_a(y) + \beta$ for $a = Y, N$. Given any partition strategy $y \in [0, 1]$, it holds that $t_Y(y) > t_N(y)$, and since $\alpha > 0$ also

$$u_Y(y) > u_N(y). \qquad (17)$$

The strategies $y = 0$ and $y = 1$ imply that one action is chosen with probability 1, so that the uniformly-random-assignment payoffs obtain. In any equilibrium y in which both actions are chosen with positive probability (i.e., $0 < y < 1$), the type $\hat{t}(y)$ is indifferent between the two actions, that is,

$$\Delta(y) = 0, \quad \text{where} \quad \Delta(y) = U_Y(y, \hat{t}(y))) - U_N(y, \hat{t}(y)). \tag{18}$$

A type t's utility gain from playing Y, given by equation 16, can be written as

$$U_Y(y,t) - U_N(y,t) = (q_Y(y) - q_N(y))(t - \hat{t}(y)).$$

In any partition equilibrium, this utility gain must be nonnegative for all $t > \hat{t}(y)$, implying

$$q_Y(y) \geq q_N(y). \tag{19}$$

Lemma 3 shows that focusing on mechanism-partition-equilibrium combinations is without loss of generality for a social planner.

Lemma 3. *For any mechanism-equilibrium combination, there exists an equivalent mechanism-partition-equilibrium combination,* $(p_1, \dots, p_{n-1}, y)$*, such that* $\Delta(y) = 0$ *and* $q_Y(y) \geq q_N(y)$*. Any mechanism-equilibrium combination with* $q_Y(y) = q_N(y)$ *is equivalent to uniformly-random assignment.*

The proof (see the Appendix) uses the fact that in a mechanism-equilibrium combination with $q_Y(y) < q_N(y)$, the agents will use a reverse partition strategy where low types play "Y" and high types play "N". An equivalent mechanism-equilibrium combination is obtained by switching the labels of the actions Y and N in the mechanism.

Given any partition-equilibrium strategy y, (19) implies that $q_Y(y) \geq 1/n$ if $y > 0$ and $q_N(y) \leq 1/n$ if $y < 1$ (cf. (11)). In other words, relative to a uniformly-random assignment, playing Y increases the probability of getting assigned the task and playing N reduces this probability. In line with our focus on partition equilibria, from now on we adopt the convention of calling the action Y "volunteering" and the action N "non-volunteering".

3.3 The planner's problem

As explained in the description of the model, the planner's objective boils down to assigning the task such that the expected type of the selected agent is maximized.

Without loss of generality, we restrict the allowed mechanism-equilibrium combinations in line with the result of Lemma 3. In particular, we only consider partition equilibria. Given any strategy y, a volunteer is selected with probability $nyq_Y(y)$ and a non-volunteer is selected with probability $n(1-y)q_N(y)$. Thus, the expected type of the selected agent is

$$W(y) = nyq_Y(y)t_Y(y) + n(1-y)q_N(y)t_N(y). \tag{20}$$

Hence, the planner's (binary-second-best) problem is to

$$\max_{p_1 \dots, p_{n-1}, y} W(y)$$

$$\begin{aligned} \text{s.t.} \quad & 0 \le p_j \le 1 \qquad (j = 1, \dots, n-1), \\ & 0 \le y \le 1, \\ & \Delta(y) = 0, \\ & q_Y(y) - q_N(y) \ge 0. \end{aligned}$$

The functions $W(y), \Delta(y), q_Y(y), q_N(y)$ of course depend on the mechanism, which is not made explicit in the notation.

Note that the planner's problem does not include a participation constraint. As implied by the lemma below, this omission is without loss of generality under the assumption that the default assignment is uniformly random. For later use, the lemma also provides conditions for a strict improvement over the uniformly-random assignment.

Lemma 4. *Consider any mechanism-equilibrium combination. Then all types are at least as well off as in the uniformly-random assignment. Let y denote the equilibrium volunteering rate. If $0 < y < 1$ and $q_Y(y) \neq q_N(y)$, then all types are strictly better off than in the uniformly-random assignment.*

The intuition behind Lemma 4 is as follows. Let $0 < y < 1$ be a partition strategy, with $q_Y(y) \ge q_N(y)$. Suppose that any agent of a given type deviates from the equilibrium by volunteering with probability y and not volunteering with probability $1 - y$. Because the agent mimics the average behavior of any other agent, she will be selected with probability $1/n$; in this event, her payoff is the same as in a uniformly-random assignment. In the complementing event that the agent is not selected, her payoff equals the equilibrium expected ability of the selected agent, which is weakly higher than the expected ability in a random assignment, and strictly so if $q_Y > q_N$. The formal proof can be found in the Appendix.

3.4 The binary-first-best benchmark

Before we solve the planner's problem in Section 4.1, we solve the problem of a planner who is not restricted by equilibrium constraints. This binary-first-best benchmark constitutes an upper bound for what can be achieved in the planner's (second-best) problem. We will show that due to the free-riding problem that is present in our model, this upper bound can typically not be achieved. The planner's *binary-first-best problem* is defined as follows:

$$\max_{p_1 \dots, p_{n-1}, y} W(y)$$

$$\begin{aligned} \text{s.t.} \quad & 0 \le p_j \le 1 \qquad (j = 1, \dots, n-1), \\ & 0 \le y \le 1. \end{aligned}$$

The interpretation is that, by setting any y, the planner has the power to make the types in $[\hat{t}(y), t_H]$ play Y and to make the types in $[t_L, \hat{t}(y)]$ play N.

This problem has a straightforward solution. Given any volunteering rate y, the welfare is maximized by maximizing the probability of assigning the task to a Y-player. This is achieved by the *any-volunteer rule* $(p_1 \dots, p_{n-1}) = (1, \dots, 1)$.

Remark 1. *The solution to the binary-first-best problem involves the any-volunteer rule. Any welfare-maximizing volunteering rate* y^* *satisfies* $0 < y^* < 1$.

The details of the proof are relegated to the Appendix. In the case $n = 2$, the first-order condition underlying Remark 1 simplifies to $\hat{t}(y^*) = \bar{t}$, that is, in the optimum the marginally volunteering type is the expected type. For $n \geq 3$, one can show that $\hat{t}(y^*) > \bar{t}$.

Social welfare and individual incentives are aligned if and only if the marginal type from the binary-first-best is indifferent between performing the task herself and letting someone else do it.

Corollary 1. *The binary second best coincides with the binary first best if and only if* $\beta + \alpha\hat{t}(y^*) = \hat{t}(y^*)$ *for a binary-first-best* y^*. *Thus, with Assumption 1 in place, the binary first-best cannot be obtained by any mechanism-equilibrium combination.*

In the heterogenous-ability case, the condition given in the corollary means that the volunteering cost is equal to 0. In the altruism case it means that $\alpha = 1$, i.e., that the other agents fully internalize the cost imposed on the selected agent. Both these limit cases are ruled out by the strict free-riding condition in Assumption 1.

The proof of Corollary 1 (see the Appendix) makes use of a formula for the welfare effect of marginally increasing the volunteering rate y. Because this formula is interesting in its own, provides good intuition, and is also used in the proof of Lemma 6, we state it as a separate result (Lemma 5). The formula holds for any binary-action rule and connects the welfare effect to $\Delta(y)$, the marginal type's utility gain from playing Y versus playing N. It captures how the misalignment between the planner's welfare goal and an agent's equilibrium condition depends on the strength of the marginal type's free-riding incentive. The free-riding incentive is proportional to the effect of the agent's own action choice on the probability of being selected, $q_Y(y) - q_N(y)$, and is proportional to the cutoff-type agent's change in utility if she performs the task versus if someone else of her type does it, $\alpha\hat{t}(y) + \beta - \hat{t}(y)$.

Lemma 5. *Consider any mechanism and any (not necessarily equilibrium) partition strategy* y. *Then*

$$\frac{\alpha}{n} W'(y) = \Delta(y) + (q_Y(y) - q_N(y))((\alpha - 1)\hat{t}(y) + \beta). \tag{21}$$

To illustrate, consider any partition-equilibrium strategy y (i.e., $\Delta(y) = 0$) with $q_Y(y) > q_N(y)$. Since the marginal type $\hat{t}(y)$ has a free-riding incentive (i.e., $(\alpha - 1)\hat{t}(y) + \beta > 0$), marginally increasing the volunteering rate increases the welfare (i.e., $W'(y) > 0$). This result reflects the very idea of free riding: individual agents have insufficient volunteering incentives from a social point of view.

3.5 Approximating the standard first-best

The binary-first-best problem maintains the restriction to binary mechanisms and partition strategies. The standard first best, in contrast, is defined without these restrictions. The solution to the standard first-best problem is to always assign the task to the agent with the highest type among all agents. Given that a continuum of types exists, this solution can obviously not be reached exactly with a binary mechanism.

Remark 2. *The first-best welfare is approximated arbitrarily closely by the binary-first-best welfare as the population becomes arbitrarily large. Formally, let y^* denote a binary-first-best volunteering rate; as $n \to \infty$, $y^* \to 0$, $ny^* q_Y(y^*) \to 1$, and $\hat{t}(y^*) \to t_H$.*

The formulas say that, in the binary first-best in a large population, the individual volunteering probability tends to 0, the probability that at least one agent volunteers tends to 1, and the expected type of the selected agent tends to the highest possible type. Remark 2 follows from the fact that in a large population, an agent with an ability close to the highest possible type exists with a probability close to 1; a detailed proof can be found in the Appendix.

4 Utilitarian optimality

In this section, we introduce a particular class of binary-action rules called threshold rules, as well as a variant of these, threshold rules with uniformly random default, and present their utilitarian optimality properties.

4.1 Solving the binary-second-best problem: threshold rules

Our first main result is that the solution to the planner's problem always involves a threshold rule (Proposition 1). A mechanism $(p_1, \ldots, p_{n-1})$ is called a *threshold rule* if there exists a number i^* $(1 \leq i^* \leq n-1)$ such that $p_j = 1$ for all $j > i^*$ and $p_j = 0$ for all $j < i^*$. We call such a rule *pure* if $p_{i^*} = 1$.

The any-volunteer rule is a threshold rule; set $i^* = 1$ and $p_{i^*} = 1$. Other threshold rules capture the idea of what we call the multiple-volunteers principle. Each agent anticipates that volunteering puts her in a lottery box together with the other volunteers if there are more than i^* volunteers in total. If the required number of i^* volunteers is not reached (or, put differently, the number of non-volunteers exceeds $n - i^*$), then all volunteers are free and the non-volunteers are put into a lottery box.

Proposition 1. *Any solution to the binary-second-best problem involves a threshold rule.*

Towards proving this, it is useful to know that the equilibrium condition can be relaxed so that it becomes an inequality. Put simply, due to the free-rider problem the planner cannot gain from artificially restricting the volunteering level.

Lemma 6. *Any solution to the binary second-best problem also solves the relaxed problem in which the constraint $\Delta(y) = 0$ is replaced by the inequality $\Delta(y) \geq 0$.*

To prove Proposition 1, we first invoke Proposition 5 below which implies that some improvement over a uniformly-random assignment is possible in any group of at least three agents (with two agents, Proposition 1 is void). Thus, from Lemma 3 the constraint $q_Y(y) \geq q_N(y)$ is not binding at the optimum. Fixing any volunteering rate y, we consider the Lagrangian first-order conditions with respect to the rule-defining probabilities p_i, where the Lagrangian multiplier of the constraint $\Delta(y) = 0$ can be assumed to be non-negative by Lemma 6. Now the stochastic independence of agents' types (as captured by the binomial distribution) leads to the optimality of a threshold rule.

The answer to the question *which threshold rule* is optimal depends on the parameters. The results in Section 4.3 imply that threshold rules with arbitrarily large i^* can be optimal as the group size n becomes large.

4.2 Binary second-best with ex-post participation constraints: rules with random default

In a threshold rule, once a volunteering agent learns the total number of volunteers, she may regret her participation if non-participation induces the uniformly-random task assignment as a default outcome. To see this, consider a partition equilibrium y in the two-volunteers rule (i.e., $i^* = 2$ and $p_{i^*} = 1$). Suppose that an agent with a non-volunteering type $t < \hat{t}(y)$ finds out that only one volunteer shows up. Then the task is assigned among the $n-1$ non-volunteers, yielding the payoff $\frac{1}{n-1}t + \frac{n-2}{n-1}u_N(y)$. But if she refuses to participate, then with probability $1/n$ the task will be assigned to the volunteer, giving the payoff $u_Y(y) > t_Y(y) > \hat{t}(y) > t$, which is also higher than $u_N(y)$.[12]

To alleviate the regret concern, we now consider ex-post participation constraints. For any type of a given agent, her ex-post payoff is defined as the expected payoff at the point when she learns the actual number of volunteers. The *binary second-best problem with ex-post participation constraints* is defined as the planner's binary second-best problem augmented by the additional requirement that, given any number of volunteers, the ex-post payoff of any type of agent in equilibrium is at least as large as her ex-post payoff in a uniformly-random assignment.

In some cases, such as those with sufficiently high volunteering costs, the solution to the binary-second-best problem with ex-post participation constraints is the uniformly-random task assignment. In such cases, any rule is optimal. In all other cases, any solution is a *threshold rule with uniformly-random default*, by which we mean a mechanism $(p_1, \ldots, p_{n-1})$ with the following property: there exists a number i^* $(1 \leq i^* \leq n-1)$ such that $p_j = 1$ for all $j > i^*$, $p_j = j/n$ for all $j < i^*$, and $p_{i^*} \geq i^*/n$. Such a rule is called "pure" if $p_{i^*} = 1$. Threshold rules with a uniformly-random default capture the multiple-volunteers principle in a similar way as the

[12] Regret of participation may also occur for a volunteer. If an agent with type t finds out she is the only volunteer in the two-volunteers rule, then someone else performs the task; the agent's payoff will be equal to the benefit $u_N(\hat{y})$. Had she refused to participate, then with probability $1/n$ she would have been assigned to the task herself and she would have gained the payoff $(1/n)(t - u_N(\hat{y}))$, which can be a strictly positive number for a sufficiently large type t. A concrete example is the setting where all other parameters are as in Figure 1, except that $\beta = 0.1$.

standard threshold rules. The only difference is that the assignment falls back to uniformly-random if not enough volunteers show up.

Proposition 2. *Suppose that the value at the solution to the binary-second-best problem with ex-post participation constraints exceeds that of a uniformly-random assignment. Then the solution involves a threshold rule with uniformly-random default.*

The proof (see the appendix) follows the lines of the proof of Proposition 1, taking account of the additional constraints as formulated below.

Lemma 7. *The binary-second-best problem with ex-post participation constraints is obtained by augmenting the binary-second-best problem with the constraints*

$$p_i \geq \frac{i}{n} \quad (i = 1, \ldots, n-1) \tag{22}$$

and

$$(p_i - \frac{i}{n}) \left(i(u_Y(y) - u_N(y)) - (u_Y(y) - \hat{t}(y)) \right) \geq 0 \quad (i = 1, \ldots, n-1). \tag{23}$$

The constraints (22) require that no matter how many agents volunteer, the probability of selecting a volunteer is at least as large as in a uniformly-random assignment. It is important to note that the constraints (22) alone are not sufficient to capture the ex-post participation constraints. Indeed, numerical examples of threshold rules with uniformly-random default can be given such that an ex-post participation constraint is violated in some partition equilibrium. But this problem cannot occur for pure rules.

Remark 3. *In any partition equilibrium of a pure threshold rule with uniformly-random default, the ex-post participation constraints are satisfied.*

The intuition behind this result is as follows. If the required number of volunteers i^* is not reached, the uniformly-random assignment obtains and the ex-post participation constraints hold with equality. If the required number of volunteers is reached, the non-volunteers receive more than the uniformly-random-assignment payoff since the task is assigned to the volunteers, who have a larger expected type than the non-volunteers. The only ex-post participation constraints that remain are those of the volunteers when the threshold i^* is reached.

Intuitively, reaching the threshold is good news for the volunteers as well, who would otherwise not have volunteered. The motivation to volunteer is due to the chance to be pivotal for assignment to the volunteers, which occurs if the number of other volunteers is equal to $i^* - 1$. Therefore, a volunteer's payoff if there are exactly i^* volunteers is larger than the uniformly-random-assignment payoff. The ex-post payoff of the marginal volunteer's type $\hat{t}(y)$ is increasing in the number of volunteers, and hence all ex-post participation constraints are satisfied in any partition equilibrium of a threshold rule with uniformly-random default that has $p_{i^*} = 1$. If $p_{i^*} < 1$, then the intuition outlined here only has the weaker implication that the ex-post participation constraints hold if there are strictly more than i^* volunteers.

4.3 Volunteering in a large population

In this section, we consider a large population of agents. We demonstrate that the first best can be approximated using a threshold rule without or with random default if the threshold is sufficiently large (Proposition 3). This result demonstrates once more the power of threshold rules. In particular, in large populations there can be no substantial efficiency gain from rules with more than two actions, or from using monetary transfers. This insight is diametrically opposed to the situation in settings of pure private values. There each agent only cares about her personal probability of getting assigned the task. In such settings, no type-specific incentives can be provided without monetary transfers, so that the assignment will always be uniformly random, whereas with monetary transfers the first best could be reached.

As a preliminary step, we characterize large-population limits of equilibrium volunteering levels for arbitrary threshold rules. Lemma 8 considers sequences of equilibria that are indexed by the population size. We show that any pure i^*-threshold rule with a sufficiently large threshold i^* has a sequence of partition equilibria along which the expected number of volunteers remains bounded away from 0 as the population becomes arbitrarily large; we derive a formula (25) for the large-population limit of the expected number of volunteers (denoted z^*). The lower bound on i^* in Lemma 8 guarantees that the right-hand side in (25) is less than 1, so that (25) has a (unique) solution. Lemma 8 also provides a formula for the limit probability that the task is assigned to a volunteer (denoted r^*), which in turn yields a formula for the limit expected type of the selected agent.

Our approach to keep i^* fixed while we let the population size grow to infinity suffices for the proof that the first best can be approximated with increasingly large thresholds, as implied by Proposition 3. A central role is played by the Poisson distribution. For any $z > 0$, let $\text{Pois}(z)(i) = e^{-z}z^i/i!$ denote the probability of the realization $i = 0, 1, \ldots$ according to the Poisson distribution with expectation z. The corresponding hazard-rate function is

$$h^{\text{Pois}(z)}(i) = \frac{\text{Pois}(z)(i)}{\sum_{j=i}^{\infty} \text{Pois}(z)(j)} = \frac{1}{i! \sum_{j=i}^{\infty} \frac{z^{j-i}}{j!}}. \tag{24}$$

Lemma 8. *Consider a threshold rule, with or without uniformly-random default, with parameter* $i^* > (\alpha t_H + \beta - t_H)/(\alpha(t_H - \bar{t}))$ *and* $p_{i^*} = 1$.

Given any sequence of partition equilibria $(\hat{y}_n)$ *defined for all population sizes* $n > i^*$*, let* $z_n = n\hat{y}_n$ *denote the corresponding expected number of volunteers.*

There exists a sequence of partition equilibria such that $\liminf_n z_n > 0$. *For any such sequence,*

$$z_n \to z^*, \quad \text{where } h^{Pois(z^*)}(i^*) = \frac{\alpha t_H + \beta - t_H}{i^* \alpha (t_H - \bar{t})}. \tag{25}$$

Let (r_n) *denote the sequence of equilibrium probabilities that the task is assigned to a volunteer,* *that is,* $r_n = n\hat{y}_n q_Y(\hat{y}_n)$ *for all* $n > i^*$*. Then*

$$r_n \to r^* \in (0,1), \quad \textit{where } r^* = \sum_{j=i^*}^{\infty} \textit{Pois}(z^*)(j).$$

The sequence of equilibrium levels of the expected type of the selected agent converges to $r^* t_H + (1-r^*)\bar{t}$*.*

The intuition behind Lemma 8 is as follows (for proof details see the Appendix). First, recall that the Poisson distribution is the limit of binomial distributions as the number of draws grows large and the expected number of successes stabilizes. For large n, the number of volunteers therefore approximately follows a Poisson distribution with mean z^*. The intuition behind the formula for z^* stated in Lemma 8 is as follows. In a large population, each agent will volunteer with a small probability, that is, the marginally volunteering type is approximately equal to t_H. Her decision to volunteer is pivotal for the population if exactly $i^* - 1$ other agents volunteer, which happens with probability

$$\text{Pois}(z^*)(i^* - 1) = \frac{i^*}{z^*}\text{Pois}(z^*)(i^*).$$

In this event, the marginal agent's decision to volunteer implies that the task will be assigned to a top type rather than to a mean type, yielding an additional benefit for every agent of approximately $\alpha(t_H - \bar{t})$.

Any given agent's payoff, however, is also affected by the probability that she is selected herself for the task if she volunteers. In a large population, this probability is approximately equal to

$$\sum_{j \geq i^*-1} \text{Pois}(z^*)(j)\frac{1}{j+1} = \frac{1}{z^*}\sum_{j\geq i^*}\text{Pois}(z^*)(j).$$

In this event, the agent's payoff will be determined by her own type rather than by the benefit provided by someone else, contributing a payoff loss of approximately $\alpha t_H + \beta - t_H$ for the marginal type. Overall, the marginal type's payoff gain from volunteering is approximately equal to

$$\frac{i^*}{z^*}\text{Pois}(z^*)(i^*)\alpha(t_H - \bar{t}) - \frac{1}{z^*}\sum_{j\geq i^*}\text{Pois}(z^*)(j)(\alpha t_H + \beta - t_H).$$

This is equal to 0 because the marginal type is indifferent between volunteering and not volunteering, yielding (25).[13]

[13] In the case $i^* = 1$, we can make a connection to the literature on of public-good provision with complete information. Bergstrom and Leo (2015) define the coordinated volunteer's dilemma as the game in which, similar to the any-volunteer rule, the task is performed by a randomly selected volunteer if and only if at least one volunteer comes forward; if nobody volunteers, then the task is not performed at all. The task has a commonly known provision cost c and a commonly known benefit b to each agent; they consider symmetric equilibria in mixed strategies. As shown by Bergstrom and Leo (2015), in their setting formulas analogous to those in Lemma 8 for the heterogenous-ability case

The formula for r^* is straightforward from the fact that the number of volunteers approximately follows a Poisson distribution with mean z^*.

We remark that, because the hazard-rate function $h^{\mathrm{Pois}(z)}(i)$ is strictly decreasing in the mean z, the expected number of volunteers, z^*, is uniquely determined and is strictly decreasing in the ratio $(\alpha t_H + \beta - t_H)/(\alpha(t_H - \bar{t}))$. Moreover, because the function $z \mapsto h^{\mathrm{Pois}(z)}(i)$ approaches the value 1 as $z \to 0$, z^* is close to 0 if the ratio is close to i^*.

The following result shows that the first-best optimal assignment can be approximated arbitrarily closely via an i^*-threshold rule if i^* is chosen sufficiently large and the population is sufficiently large. This result is important because it shows that binary mechanisms, although being very simple with just two possible actions for each agent, are sufficient to approximate the first best in a large population. The reason a binary mechanism *may be* good enough is that the information to be extracted from the agents is binary as well: each agent is essentially asked whether or not her type is close to the highest feasible type. The nonobvious feature of the i^*-threshold rule with large i^* is that in equilibrium it becomes almost certain that at least i^* volunteers will come forward if the population is sufficiently large.

Proposition 3. *As the population size converges to infinity, any corresponding sequence of welfare-optimal threshold levels also converges to infinity, and the resulting second-best welfare level converges to the first-best welfare level. Formally,* $\lim_{i^* \to \infty} r^* = 1$*, where* r^* *is defined in Lemma 8.*

The proof relies on the fact that, by the equilibrium condition (25), an agent's pivotality probability and her probability of getting assigned the task if she volunteers must remain comparable in size as the threshold i^* grows large; i.e., the ratio of the two probabilities converges to a number strictly between 0 and infinity as i^* tends to infinity. This implies that the expected number of volunteers, z^*, will remain sufficiently (just a little) above i^* such that at large thresholds i^* it becomes almost certain that the threshold will be reached in equilibrium. The proof uses the Chernoff bounds for tail probabilities of Poisson random variables.

5 Robust improvement-property

We now move from optimality to improvement considerations. Which threshold rules allow for some improvement over a uniformly random task assignment? In the spirit of Wilson's ideal of detail-freedom, a social planner will find it reassuring to implement a rule that allows for a robust improvement in the sense that the improvement occurs across a large range of parameters.

(i.e., $\alpha = 1$ and $\beta = c$) hold if $t_H - \bar{t}$ is replaced by b. This is intuitive because, in our model, the social benefit of the volunteering decision of the marginal type is that the task is done at highest quality, t_H, rather than average quality, $\bar{t}$. The lower bound for i^* in Lemma 8 is then satisfied for the any-volunteer rule ($i^* = 1$) if $1 > c/b$, implying that a sequence of equilibria with a strictly positive limit for the expected number of volunteers exists if $c < b$.

5.1 Breakdown of volunteering with random-default rules

One may conjecture that a robust improvement is impossible because in a setting with large volunteering costs or too little altruism there would be no volunteering at all. In this section, we confirm this conjecture for the threshold rules with uniformly-random default (Proposition 4). This negative conclusion sets the stage for our result in the next section that yields the *opposite* conclusion for any non-extreme threshold rule *without* uniformly-random default: any such rule *always* allows for an improvement over a uniformly random assignment!

Proposition 4. *Consider any threshold rule with uniformly-random default that is different from the uniformly-random-assignment rule,* $(p_1, \ldots, p_{n-1}) \neq (1/n, \ldots, (n-1)/n)$*. Let* i^* *be the smallest* i *such that* $p_i > i/n$*.*

For all $y \in (0, 1]$*, it holds that* $q_Y(y) > q_N(y)$*, implying that any equilibrium is a partition equilibrium.*

If $t_H + (i^* - 1)(\alpha t_H + \beta) \leq i^*(\alpha\bar{t} + \beta)$*, then in the unique equilibrium nobody volunteers. Otherwise, there exists an equilibrium in which every type of agent is strictly better off than in the uniformly-random assignment.*

The reason for the possible breakdown of volunteering is easiest to explain for the any-volunteer rule (i.e., $i^* = 1$, $p_{i^*} = 1$). Consider an agent of highest type who believes that nobody else will volunteer. Switching her action from non-volunteering to volunteering raises the probability that she herself gets assigned the task by $1 - 1/n$. At the same time, the switch reduces, by the same amount, the probability that a non-volunteer other than herself is selected. Thus, the agent faces an equal-probability tradeoff between the payoff from volunteering herself, t_H, and the payoff from letting somebody else do the job, $\alpha\bar{t} + \beta$. Thus, non-volunteering is optimal if $t_H \leq \alpha\bar{t} + \beta$ and otherwise volunteering is optimal. A more elaborate argument shows that under the same condition, for *any* volunteering rate the marginal type prefers to not volunteer. Thus, everybody non-volunteering constitutes the unique equilibrium. The arguments generalize to other threshold rules with uniformly-random default, in particular rules with $i^* \geq 2$. The details of the proof are relegated to the Appendix.

In the heterogeneous-ability interpretation of our model, the no-volunteering condition in Proposition 4 translates to $i^*(\theta_H - \bar{\theta}) \leq c$, where θ_H denotes the highest skill type, $\bar{\theta}$ denotes the average skill type, and c denotes the volunteering cost. Hence, in the heterogeneous-ability case of our model, volunteering breaks down if and only if the volunteering cost is sufficiently high or all agents have sufficiently similar abilities. In the altruism interpretation, the condition in Proposition 4 is $-t_H \geq \frac{\alpha i^*}{1-\alpha}(-\bar{t})$, where $-t_H$ denotes the lowest provision cost, $-\bar{t}$ denotes the average provision cost, and $0 < \alpha < 1$ is the level of altruism. Hence, in this case volunteering breaks down if and only if the level of altruism is sufficiently close to 0.

5.2 Improvement over the uniformly-random assignment

The main result in this section is Proposition 5, in which we show that any threshold rule that is sufficiently different from the any-volunteer rule (and its mirror rule, the all-volunteer rule), *always* has an equilibrium that provides a strict improvement over the uniformly-random assignment. We call these threshold rules *non-extreme.*

The robust-improvement property of non-extreme threshold rules marks a crucial advantage of these rules over using a threshold rule with a uniformly-random default. By committing to select a non-volunteering agent for the task if the threshold is not reached, the designer effectively creates a cost of non-volunteering that always keeps up some volunteering activity. In equilibrium, still all non-volunteering agent types are strictly better off than in a uniformly random assignment.

A threshold rule $(p_1, \ldots, p_{n-1})$ is called *non-extreme* if the assignment probability to a volunteer is below uniform randomness if there is a single volunteer (i.e., $p_1 < 1/n$), and the assignment probability to a non-volunteer is below uniform randomness if there is a single non-volunteer (i.e., $p_{n-1} > 1 - 1/n$). This condition is satisfied for all threshold rules with $2 \leq i^* \leq n - 2$. A non-extreme threshold rule exists if and only if $n > 2$. Proposition 5 shows that any non-extreme threshold rule always allows for a strict improvement over the uniformly-random assignment. Note that the result holds for arbitrarily large volunteering-cost levels and for altruism levels that are arbitrarily close to 0.

Proposition 5. *Any non-extreme threshold rule has an equilibrium in which every type of agent is strictly better off than in a uniformly-random assignment. A partition-equilibrium strategy with the strict-improvement property is given by the maximal element $\hat{y}$ of the set $\{y \in [0, 1] \mid \Delta(y) = 0\}$.*

The contribution of Proposition 5 is not the existence of some equilibrium with a positive volunteering rate, but of a partition-equilibrium strategy $\hat{y}$ that satisfies the strict-improvement conditions, $0 < \hat{y} < 1$ and $q_Y(\hat{y}) > q_N(\hat{y})$. It is straightforward to see that in any equilibrium (partition or else) there is some volunteering. Indeed, the no-volunteering strategy $y = 0$ is never an equilibrium: an agent with a type $t \approx t_L$ (sic!) who expects no other volunteers to come forward will be selected with probability p_1 if she volunteers and with probability $1/n$ otherwise, inducing a payoff gain from volunteering that is equal to $(p_1 - 1/n)(t - \alpha\bar{t} - \beta)$, which is > 0 by Assumption 1.

Figure 1 helps to understand the proof of Proposition 5. Consider the function q_Y, which captures an agent's probability of getting assigned the task if she volunteers. A crucial point of comparison is the agent's ex-ante selection probability $1/n$, marked as a horizontal line in the figure. For any volunteering rate $y \in (0, 1)$ of the other agents, volunteering increases the personal selection probability (i.e., $q_Y(y) > q_N(y)$) if and only if $q_Y(y) > 1/n$. To prove Proposition 5, we show that $q_Y(\hat{y}) > 1/n$ at the last point $\hat{y}$ where the function Δ crosses the y-axis.

Intuitively, in a non-extreme threshold rule, by volunteering an agent reduces her personal selection probability if the others' volunteering rate is sufficiently low, and otherwise increases it. Formally, we establish the existence of a unique volunteering rate $\check{y} \in (0, 1)$ where the function q_Y crosses the $1/n$-line from below. At $\check{y}$, because an agent's action has no impact on her own selection probability, the benefit from volunteering, $\Delta(\check{y})$, is proportional to $h_Y - h_N - 1/n$, the agent's effect on the probability that a volunteer other than herself is selected. Hence, once we

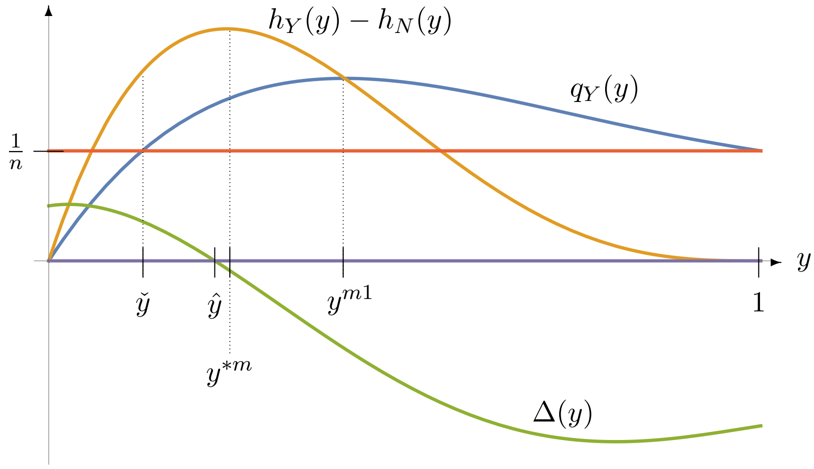

Figure 1: An example of a task assignment problem. There are $n = 5$ agents. The type of each agent is uniformly distributed on $[-1, 0]$. The preference parameters are $\alpha = \beta = 1$. The diagram shows several functions of the volunteering rate y, for the case of the threshold rule that requires two volunteers (i.e., $i^* = 2$ and $p_{i^*} = 1$). The function $h_Y(y) - h_N(y)$ captures the impact of an agent's switch from non-volunteering to volunteering on the probability (computed from the switching agent's point of view) that the task gets assigned to a volunteer. The function $q_Y(y)$ captures the probability an agent assigns to the event of being selected if she volunteers. The function $\Delta(y)$ captures an agent's payoff gain from volunteering. The volunteering rate $\hat{y}$ is a partition-equilibrium strategy that satisfies the strict-improvement conditions stated in Lemma 4.

establish that $h_Y(\check{y}) - h_N(\check{y}) > 1/n$, we can conclude that $\hat{y} > \check{y}$, hence $q_Y(\hat{y}) > 1/n$, and we are done.

How do we prove that q_Y crosses the $1/n$-line only once, from below, as claimed above? If nobody else volunteers (i.e., $y = 0$), a volunteering agent expects to be selected with probability p_1. Thus, $q_Y(0) < 1/n$ for any non-extreme threshold rule, that is, at the point 0 the function q_Y is below the $1/n$-line. On the other hand, if everybody else volunteers (i.e., $y = 1$), then volunteering herself puts the agent in a pot with everybody, that is, $q_Y(1) = 1/n$. Given this, it is sufficient to show that q_Y is strictly quasi-concave, that is, is first strictly increasing and then strictly decreasing. But this is immediate from Lemma 2 and the following result.

Lemma 9. *For any non-extreme threshold rule, there exists a $y^{m1} \in (0, 1)$ such that $h_Y(y) - h_N(y) > q_Y(y)$ for all $y < y^{m1}$ and $h_Y(y) - h_N(y) < q_Y(y)$ for all $y > y^{m1}$.*

Note that $q_Y(y^{m1}) > 1/n$, implying $\check{y} < y^{m1}$.[14] Hence, Lemma 9 implies $h_Y(\check{y}) - h_N(\check{y}) > q_Y(\check{y}) = 1/n$, establishing the last remaining claim in the text above.

Lemma 9 says that if the volunteering rate is below some threshold, then by volunteering an agent increases the probability that a volunteer other than herself gets assigned the task, and otherwise she decreases this probability. The proof of this single-crossing property relies on another lemma, Lemma 15, that establishes the quasi-concavity of the function $h_Y - h_N$. The quasi-concavity relies on the special form this function has for a threshold rule. All these details are relegated to the appendix.

[14]While in Figure 1 it is also true that $\hat{y} < y^{m1}$, this inequality can be reversed, such as when $\beta = 0.1$ and all other parameters are as in Figure 1.

Proposition 5 talks about one particular equilibrium, raising questions about alternative equilibria. Recall that any strategy σ gives rise to a volunteering rate $y(\sigma)$. Any equilibrium strategy σ with $q_Y(y(\sigma)) > q_N(y(\sigma))$ yields a partition equilibrium. Recall that $0 < y(\sigma) < 1$, implying $\Delta(y(\sigma)) = 0$. If $\Delta'(y(\sigma)) < 0$, then σ is *dynamically stable*: if an agent expects others to volunteer at a rate y slightly above $y(\sigma)$, then $\Delta(y) < 0$, implying that her best response is to volunteer at a rate slightly below y, and vice versa for volunteering rates slightly below $y(\sigma)$. Any equilibrium σ with $q_Y(y(\sigma)) = q_N(y(\sigma))$ is dynamically unstable: if an agent expects others to volunteer at a rate y slightly above $y(\sigma)$, then $q_Y(y) > q_N(y)$ and $\Delta(y) > 0$ (cf. Figure 1), implying that her best response is a partition strategy in which she volunteers at a rate even higher than y.

Corollary 2. *For any non-extreme threshold rule, the equilibrium described in Proposition 5 is dynamically stable. In any dynamically stable equilibrium, every type of agent is strictly better off than in a uniformly-random assignment.*

Proof. The claims concerning equilibria σ with $q_Y(y(\sigma)) = q_N(y(\sigma))$ or $q_Y(y(\sigma)) > q_N(y(\sigma))$ follow from the explanations given above. An equilibrium σ with $q_Y(y(\sigma)) < q_N(y(\sigma))$ has the same properties as an equilibrium with $q_Y > q_N$ in the rule that is obtained by switching the names of the actions Y and N. □

6 Assigning a pleasant task

So far we have maintained Assumption 1, which means that the task is unpleasant. With the opposite assumption, our model is suited for applications in which there are excessive volunteering incentives. Instead of volunteering for an unpleasant task, agents who choose the message "Y" propose themselves for a promotion or a prize for some achievement. Agents then prefer to be selected over another agent with the same merit, but they also care about the merit of the agent who is selected.[15]

To model the problem of assigning a pleasant task, we hence keep the condition that the performer's utility is congruent with the other agents' utilities, but we assume that agents of all types prefer doing the task themselves rather than letting someone of their own type provide the service. The latter condition means that getting people to refrain from volunteering can be difficult.

Assumption 1'.

$$\alpha > 0 \quad \textit{and} \quad t > \alpha t + \beta \textit{ for all } t \in [t_L, t_H]. \tag{26}$$

In the heterogenous-ability case, Assumption 1' means that the cost $c < 0$. In the altruism case, it means that only strictly positive types are possible, $t_L > 0$.

[15] Another example is the setting of Li, Rantakari, and Yang (2016), in which two agents would like their own project to be chosen, but also have a common interest in the quality of the chosen project. The setting of Bhaskar and Sadler (2019) also fits in with $\beta = 0$, except that they also allow for a non-provision of the public good.

Assumption 1' is interesting beyond the pleasant-task interpretation. We can interpret the model as the problem of assigning $n-1$ unpleasant tasks among n agents (where the spared agent's private information determines everybody's payoff).

The following paragraphs describe how our results continue to hold with some modifications. This streamlined exposition does not always follow the order of the exposition in the main part and does not explicitly state all results again. Those that are stated always receive the number of the corresponding results in the main part. The proofs are mostly analogous to the proofs of the corresponding results in the main part and can be found in the appendix.

Lemma 1'. *The only anonymous ex-post incentive-compatible (EPIC) task-assignment rule is the uniformly-random rule.*

We will use the same expected-utility formulas as in the main part, (2) and (3). Further, note that the formulas of the selection-probability functions and their properties do not depend on the utility function. We also still have the equivalence result (Lemma 3): For any mechanism-equilibrium combination, there exists an equivalent mechanism-partition-equilibrium combination, $(p_1, \ldots, p_{n-1}, y)$, such that $\Delta(y) = 0$ and $q_Y(y) \geq q_N(y)$. Any mechanism-equilibrium combination with $q_Y(y) = q_N(y)$ is equivalent to uniformly-random assignment. This result needs no assumptions on α and β. It is basically the result that only monotonically increasing functions can be implemented. Recall that we use the term "volunteering" for the action Y and "non-volunteering" for N. Given that we now consider the assignment of a pleasant task, a Y-playing agent may also be called a "keen" agent.

Some other results only rely on the assumption that $\alpha > 0$ and do not use the assumption that the task is unpleasant. For example, the proof of Lemma 4 still holds without any modification. Hence, we know also in this case that for any mechanism-equilibrium combination, all types are at least as well off as in the uniformly-random assignment. Moreover, if the equilibrium volunteering rate y has $0 < y < 1$ and $q_Y(y) \neq q_N(y)$, then all types are strictly better off than in the uniformly-random assignment.

Since $\alpha > 0$, maximizing welfare is still equivalent to maximizing the type of the selected agent. The planner's *binary-first-best problem* is defined and solved as in the main part: The solution to the binary-first-best problem involves the any-volunteer rule, any welfare-maximizing volunteering rate y^* satisfies $0 < y^* < 1$, and the first-best welfare is approximated arbitrarily closely by the binary-first-best welfare as the population becomes arbitrarily large.

The planner's (binary-second-best) problem is defined as in the main part. The binary-first-best benchmark constitutes an upper bound for what can be achieved in the planner's problem. Due to the excessive volunteering incentives that are present in this model, this upper bound can typically not be achieved. This can be seen from the formula for the welfare effect of marginally increasing the volunteering rate y that was derived in the main part (Lemma 5):

$$\frac{\alpha}{n} W'(y) = \Delta(y) + (q_Y(y) - q_N(y))((\alpha - 1)\hat{t}(y) + \beta).$$

To illustrate, consider any partition-equilibrium strategy y (i.e., $\Delta(y) = 0$). If an agent's task-assignment probability is not independent of her action (i.e., $q_Y(y) > q_N(y)$) and the marginal type $\hat{t}(y)$ actually has an excessive volunteering incentive (i.e., $(\alpha - 1)\hat{t}(y) + \beta < 0$), then marginally increasing the volunteering rate decreases the welfare (i.e., $W'(y) < 0$). That is, from a social point of view it would be good to curb the volunteering activity a bit—the volunteering incentives are excessive at the margin.

This formula can again be used to prove that social welfare and individual incentives are aligned if and only if the marginal type from the binary-first-best is indifferent between performing the task herself and letting someone else do it (Corollary 1), only that now the condition $\beta = \hat{t}(y^*)(1 - \alpha)$ cannot be satisfied for a binary-first-best y^* because of Assumption 1' instead of Assumption 1.

Solving the binary-second-best problem: maximum-threshold rules

The main result of this section is that the solution to the planner's problem always involves a maximum-threshold rule (Proposition 1'). A mechanism $(p_1, \ldots, p_{n-1})$ is called a *max(imum)-threshold rule* if there exists a number i^* ($1 \leq i^* \leq n-1$) such that $p_j = 1$ for all $1 \leq j < i^*$ and $p_j = 0$ for all $n - 1 \geq j > i^*$. With a maximum-threshold rule, the task is awarded to a keen individual if there are sufficiently *few* keen individuals.[16]

A max-threshold rule $(p_1, \ldots, p_{n-1})$ is called *non-extreme* if the assignment probability to a volunteer is above uniform randomness if there is a single volunteer (i.e., $p_1 > 1/n$), and the assignment probability to a non-volunteer is above uniform randomness if there is a single non-volunteer (i.e., $p_{n-1} < 1 - 1/n$). This condition is satisfied for all max-threshold rules with $2 \leq i^* \leq n - 2$. Proposition 5' shows that any non-extreme max-threshold rule always allows for a strict improvement over the uniformly-random assignment. To prove this, we make the additional technical assumption here that the density f is at least n times differentiable.

Proposition 5'. *Any non-extreme max-threshold rule has a (dynamically stable) equilibrium in which every type of agent is strictly better off than in a uniformly-random assignment.*

The contribution of Proposition 5' is not the existence of some equilibrium with a volunteering rate strictly below 1, but of a partition-equilibrium strategy $\hat{y}$ that satisfies the strict-improvement conditions, $0 < \hat{y} < 1$ and $q_Y(\hat{y}) > q_N(\hat{y})$. It is straightforward to see that in any equilibrium (partition or else) some types will refrain from volunteering. Indeed, the all-volunteering strategy $y = 1$ is never an equilibrium: an agent with a type $t \approx t_H$ who expects all other agents to volunteer will be selected with probability $1 - p_{n-1}$ if she refrains from volunteering and with probability $1/n$ otherwise, inducing a payoff gain from volunteering that is equal to $(p_{n-1} - (1 - 1/n))(t - \alpha\bar{t} - \beta)$, which is < 0 by Assumption 1'.

To solve the planner's problem, it is again useful to show first that the equilibrium condition can be relaxed so that it becomes an inequality. This result carries over from the unpleasant

[16] One should nevertheless keep in mind that if no keen individuals show up, then—as in any rule—the task is allocated to a non-keen individual. Similarly, if all individuals are keen, then the task is naturally allocated to a keen individual. Hence, in contrast to the case of a threshold rule in the unpleasant task case, the probability that the task is assigned to a volunteer is not monotonically increasing in the number of volunteers. The analog of that monotonicity here is that the probability of a keen agent being assigned the task is monotonically increasing in the number of agents who refrain from volunteering.

task case with opposite sign: Due to the excessive-volunteering problem the planner cannot gain from increasing the volunteering level beyond the equilibrium level.

Lemma 6'. *Any solution to the binary second-best problem also solves the relaxed problem in which the constraint* $\Delta(y) = 0$ *is replaced by the inequality* $\Delta(y) \leq 0$.

To solve the planner's problem, we first invoke Proposition 5' which implies that some improvement over a uniformly-random assignment is possible in any group of at least three agents (with two agents, any rule is a max-threshold rule). Thus, the constraint $q_Y(y) \geq q_N(y)$ is not binding at the optimum. Fixing any volunteering rate y, we consider the Lagrangian first-order conditions with respect to the rule-defining probabilities p_i, where the Lagrangian multiplier of the constraint $\Delta(y) = 0$ can be assumed to be non-positive by Lemma 6'. Now the stochastic independence of agents' types leads to the optimality of a maximum threshold rule.

Proposition 1'. *Any solution to the binary-second-best problem involves a maximum-threshold rule.*

Binary second-best with ex-post participation constraints: rules with random default

In a max-threshold rule, once a volunteering agent learns the total number of volunteers, she may regret her participation if non-participation induces the uniformly-random task assignment as a default outcome. To see this, consider a partition equilibrium $\hat{y}$ in the max-one-volunteers rule $(p_1, \dots, p_{n-1}) = (1, 0, \dots, 0)$. If an agent with type t finds out she is one of two volunteers, then a non-volunteer performs the task; the agent's payoff will be equal to the benefit $u_N(\hat{y})$. But if she refuses to participate, then her expected payoff will be $(1/n)t + (1/n)u_Y(\hat{y}) + (n-2)/n \; u_N(\hat{y})$, which is an improvement because $u_Y(\hat{y}) > u_N(\hat{y})$ and $t > \hat{t}(\hat{y}) > t_N(\hat{y}) > u_N(\hat{y})$.

Let us thus consider the binary second-best problem with ex-post participation constraints as defined in the main part. In some cases, such as those with sufficiently strongly negative volunteering costs, the solution to the binary-second-best problem with ex-post participation constraints is the uniformly-random task assignment. In such cases, any rule is optimal. In all other cases, any solution is a *max-threshold rule with uniformly-random default*, by which we mean a mechanism $(p_1, \dots, p_{n-1})$ with the following property: there exists a number i^* $(1 \leq i^* \leq n-1)$ such that $p_j = 1$ for all $j < i^*$, $p_j = j/n$ for all $j > i^*$, and $p_{i^*} \geq i^*/n$. Such a rule is called "pure" if $p_{i^*} = 1$. The only difference to a standard max-threshold rule is that the assignment falls back to uniformly-random if too many volunteers show up.

Proposition 2'. *Suppose that the value at the solution to the binary-second-best problem with ex-post participation constraints exceeds that of a uniformly-random assignment. Then the solution involves a max-threshold rule with uniformly-random default.*

The proof (see the appendix) follows the lines of the proof of Proposition 1', taking account of the additional constraints as formulated below.

Lemma 7'. *The binary-second-best problem with ex-post participation constraints is obtained by augmenting the binary-second-best problem with the constraints*

$$p_i \geq \frac{i}{n} \quad (i = 1, \ldots, n-1) \tag{27}$$

and

$$(p_i - \frac{i}{n})\left((n-i)(u_Y - u_N) - (\hat{t}(y) - u_N)\right) \geq 0. \quad (i = 1, \ldots, n-1). \tag{28}$$

It is important to note the existence of the constraints (28). For pure rules, however, the ex-post participation constraints are automatically satisfied.

Remark 3'. *In any partition equilibrium of a pure max-threshold rule with uniformly-random default, the ex-post participation constraints are satisfied.*

Breakdown of volunteering with random-default rules

One may conjecture that a robust improvement is impossible because in a setting with large negative volunteering costs or too little altruism all agent types would volunteer. Here we confirm this conjecture for the max-threshold rules with uniformly-random default.

Proposition 4'. *Consider any max-threshold rule with uniformly-random default that is different from the uniformly-random-assignment rule, $(p_1, \ldots, p_{n-1}) \neq (1/n, \ldots, (n-1)/n)$. Let i^* be the largest $i \leq n-1$ such that $p_i > i/n$.*

For all $y \in [0,1)$, it holds that $q_Y(y) > q_N(y)$, implying that any equilibrium is a partition equilibrium.

If $t_L - u_L < (n-i^*)\alpha(\bar{t} - t_L)$, then there exists an equilibrium in which every type of agent is strictly better off than in the uniformly-random assignment. If there exists an equilibrium in which everybody volunteers, then it is not dynamically stable.

If $t_L - u_L > (n-i^*)\alpha(\bar{t} - t_L)$ then there exists a dynamically stable equilibrium in which everybody volunteers. If $\hat{t} - u_N \geq (n-i^*)\alpha(t_Y - t_N)$ for all $y \in [0,1]$, then this is the unique equilibrium.

In the heterogeneous-ability interpretation of our model, if the type distribution is uniform, then the condition for a dynamically stable everybody-volunteering equilibrium in Proposition 4' translates to $(n-i^*)(\bar{\theta} - \theta_L) \leq -c$, where θ_L denotes the lowest skill type, $\bar{\theta}$ denotes the average skill type, and c denotes the volunteering cost. If F is uniform, this condition already implies the condition for uniqueness of this equilibrium. For all distributions F, the uniqueness condition is satisfied if the volunteering benefit is sufficiently high or all agents have sufficiently similar abilities. In the altruism interpretation, the condition for a dynamically stable equilibrium with everybody volunteering in Proposition 4' is $t_L(1-\alpha) \geq (n-i^*)\alpha(\bar{t} - t_L)$, where t_L denotes the lowest provision benefit, $\bar{t}$ denotes the average provision benefit, and $0 < \alpha < 1$ is the level of altruism. If F is uniform, uniqueness is already implied. Else, it is satisfied for α low enough. Hence, no screening of types will happen in this case if and only if the level of altruism is sufficiently close to 0.

Recall that we arrived at the opposite conclusion for any non-extreme max-threshold rule *without* uniformly-random default: any such rule always allows for an improvement over a uniformly random assignment. This robust-improvement property of non-extreme max-threshold rules marks a crucial advantage of these rules over using a max-threshold rule with a uniformly-random default. By committing to select a non-volunteering agent for the task if the threshold number of volunteers is surpassed, the designer effectively creates a benefit of non-volunteering that always repels some types from volunteering. In equilibrium, still all agent types are strictly better off than in a uniformly random assignment.

Volunteering in a large population

In this section, we consider a large population of agents. We demonstrate that the first best can be approximated using a max-threshold rule without or with random default if the threshold is sufficiently large (Proposition 3'). As an intermediate step, we characterize large-population limits of equilibrium volunteering levels for arbitrary max-threshold rules (Lemma 8').

To simplify the exposition, we only consider "pure" max-threshold rules, that is, we assume that $p_{i^*} = 1$, where $i^* \geq 1$ is the threshold.

Lemma 8' considers sequences of equilibria that are indexed by the population size. We show that any pure i^*-max-threshold rule with a sufficiently large threshold i^* has a sequence of partition equilibria along which the expected number of volunteers remains below i^* as the population becomes arbitrarily large; we derive a formula (30) for the large-population limit of the expected number of volunteers (denoted z^*). The lower bound on i^* in Lemma 8' guarantees that (30) has a solution. Lemma 8' also provides a formula for the limit probability that the task is assigned to a volunteer (denoted r^*), which in turn yields a formula for the limit expected type of the selected agent.

Our approach to keep i^* fixed while we let the population size grow to infinity suffices for the proof that the first best can be approximated with increasingly large thresholds, as implied by Proposition 3'.

A central role is played by the Poisson distribution. For any $z > 0$, let $\text{Pois}(z)(i) = e^{-z} z^i / i!$ denote the probability of the realization $i = 0, 1, \dots$ according to the Poisson distribution with expectation z.

Lemma 8'. *Consider a max-threshold rule, with or without uniformly-random default, with parameter i^* so large that*[17]

$$i^* e^{-i^*} \left(\frac{(i^*)^{i^*}}{i^*!} - 1 \right) \quad > \quad \frac{t_H - (\alpha t_H + \beta)}{\alpha (t_H - \bar{t})}. \tag{29}$$

Given any sequence of partition equilibria $(\hat{y}_n)$, let $z_n = n \hat{y}_n$ denote the corresponding expected number of volunteers.

[17]Note that from Stirling's formula it is straightforward that the left-hand side in (29) tends to infinity, showing that the condition is satisfied for all sufficiently large i^*.

There exists a sequence of partition equilibria such that $\liminf_n z_n \leq i^*$. *For any limit point* z^* *of such a sequence,*

$$\frac{Pois(z^*)(i^*) - Pois(z^*)(0)}{\frac{1}{z^*}\sum_{j=1}^{i^*} Pois(z^*)(j)} = \frac{t_H - (\alpha t_H + \beta)}{\alpha(t_H - \bar{t})}. \tag{30}$$

Let (r_n) *denote the sequence of equilibrium probabilities that the task is assigned to a volunteer, that is,* $r_n = n\hat{y}_n q_Y(\hat{y}_n)$. *Then*

$$r_n \to r^* \in (0,1), \quad \textit{where } r^* = \sum_{j=1}^{i^*} Pois(z^*)(j).$$

The sequence of equilibrium levels of the expected type of the selected agent converges to $r^* t_H + (1 - r^*)\bar{t}$.

The intuition behind Lemma 8' is as follows. First, recall that the Poisson distribution is the limit of binomial distributions as the number of draws grows large and the expected number of successes stabilizes. For large n, the number of volunteers therefore approximately follows a Poisson distribution with mean z^*. The intuition behind the formula for z^* stated in Lemma 8' is as follows. In a large population, each agent will volunteer with a small probability, that is, the marginally volunteering type is approximately equal to t_H. Her decision to volunteer is positively pivotal for the population if 0 other agents volunteer, and is negatively pivotal if i^* other agents volunteer, yielding a net negative pivotality probability of

$$\text{Pois}(z^*)(0) - \text{Pois}(z^*)(i^*).$$

Here, the marginal agent's decision to volunteer reduces the probability that the task will be assigned to a top type rather than to a mean type, yielding a payoff loss for every agent of approximately $\alpha(t_H - \bar{t})$.

Any given agent's payoff, however, is also affected by the probability that she is selected herself for the task if she volunteers. In a large population, this probability is approximately equal to

$$\frac{1}{z^*}\sum_{j=1}^{i^*} \text{Pois}(z^*)(j).$$

In this event, the agent's payoff will be determined by her own type rather than by the benefit provided by someone else, contributing a payoff gain of approximately $t_H - (\alpha t_H + \beta)$ for the marginal type. Overall, the marginal type's payoff gain from volunteering is approximately equal to

$$\frac{1}{z^*}\sum_{j=1}^{i^*} \text{Pois}(z^*)(j)(t_H - (\alpha t_H + \beta)) - (\text{Pois}(z^*)(i^*) - \text{Pois}(z^*)(0))\,\alpha(t_H - \bar{t}).$$

This is equal to 0 because the marginal type is indifferent between volunteering and not volunteering, yielding (30).

The formula for r^* is straightforward from the fact that the number of volunteers approximately follows a Poisson distribution with mean z^*.

The following result shows that the first-best optimal assignment can be approximated arbitrarily closely via an i^*-threshold rule if i^* is chosen sufficiently large and the population is sufficiently large. This result is important because it shows that binary mechanisms, although being very simple with just two possible actions for each agent, are sufficient to approximate the first best in a large population. The reason a binary mechanism *may be* good enough is that the information to be extracted from the agents is binary as well: each agent is essentially asked whether or not her type is close to the highest feasible type. The nonobvious feature of the i^*-threshold rule with large i^* is that in equilibrium it becomes almost certain that at least i^* volunteers will come forward if the population is sufficiently large.

Proposition 3'. *As the population size converges to infinity, any corresponding sequence of welfare-optimal threshold levels also converges to infinity, and the resulting second-best welfare level converges to the first-best welfare level. Formally,* $\lim_{i^* \to \infty} r^* = 1$*, where* r^* *is defined in Lemma 8'.*

The proof is shorter than the proof of the corresponding result for an unpleasant task. This is because we know by construction that $z^* \leq i^*$, whereas in the unpleasant-task situation the corresponding opposite inequality had to be proven first.

Conclusion

If a task is to be assigned among a group of agents, the multiple-volunteers principle turns out to be a powerful tool for improving welfare with binary-action mechanisms. We expect the principle to extend to settings in which multiple tasks must be assigned: one should typically require a number of volunteers that is larger than the number of tasks. While we think that binary-action rules are a natural starting point due to their practicality, the kind of preferences that we have modelled clearly deserve attention beyond binary-action rules, including environments with monetary transfers.

Appendix: Proofs

Proof of Lemma 1. Using the notation of Bhaskar and Sadler (2019), consider any anonymous EPIC direct-mechanism rule $Q = (Q_1, \ldots, Q_n)$, where $Q_i(t)$ denotes the probability that the task is assigned to agent i at type profile $t = (t_1, \ldots, t_n)$, and $\sum_{j=1}^{n} Q_j(t) = 1$. EPIC means that, for all i, t_i, and t_{-i},

$$t_i Q_i(t_i, t_{-i}) + \sum_{j \neq i} (\alpha t_j + \beta) Q_j(t_i, t_{-i}) \geq t_i Q_i(t_i', t_{-i}) + \sum_{j \neq i} (\alpha t_j + \beta) Q_j(t_i', t_{-i}).$$

By standard revealed-preference arguments, this implies that Q_i is weakly increasing in t_i.

Anonymity means that $Q_{\rho(i)}(t^{\rho}) = Q_i(t)$ for any permutation ρ of the set $\{1, \ldots, n\}$, any agent i, and any type profile t, where t^{ρ} denotes the permuted type profile. In particular, for any i and j, using the permutation ρ that switches i and j, one sees that, for all type profiles t,

$$Q_i(t) = Q_j(t) \quad \text{if } t_i = t_j. \tag{31}$$

We do a proof by induction. For all $k = 1, \ldots, n$, consider the following statement A_k: for all type profiles t such that the highest $n - k + 1$ components are identical, we have $Q_i(t) = 1/n$ for all $i = 1, \ldots, n$.

The statement A_1 is true by (31). We will show $A_k \Rightarrow A_{k+1}$ for all $k < n$.

Assume A_k. To prove A_{k+1}, consider any type profile t such that the highest $n-k$ components are identical. Without loss of generality, we consider an ordered type profile, $t_1 \leq \cdots \leq t_n$. Thus, there exists a type $\hat{t}$ such that $t_{k+1} = \cdots = t_n = \hat{t}$. The goal is to show that

$$Q_i(t) \quad = \quad \frac{1}{n} \quad \text{for all } i. \tag{32}$$

From A_k,

$$Q_i(t_1, \ldots, t_{j-1}, \hat{t}, t_{j+1}, \ldots, t_n) \quad = \quad \frac{1}{n} \quad \text{for all } i \text{ and all } j \leq k. \tag{33}$$

It remains to consider type profiles t such that

$$t_j \quad < \quad \hat{t} \quad \text{for all } j \leq k. \tag{34}$$

Because Q_j is weakly increasing in agent j's type,

$$Q_j(t) \quad \leq \quad \frac{1}{n} \quad \text{for all } j \leq k. \tag{35}$$

EPIC implies that agent k with type $\hat{t}$ does not gain from reporting type t_k instead. This together with (33) implies that

$$\hat{t}\frac{1}{n} + \sum_{j<k} (\alpha t_j + \beta) \frac{1}{n} + \sum_{j>k} (\alpha \hat{t} + \beta) \frac{1}{n} \quad \geq \quad \hat{t} Q_k(t) + \sum_{j<k} (\alpha t_j + \beta) Q_j(t) + \sum_{j>k} (\alpha \hat{t} + \beta) Q_j(t).$$

Subtracting $\alpha\hat{t}+\beta = \sum_{j=1}^{n} Q_j(t)(\alpha\hat{t}+\beta)$ on both sides yields

$$(\hat{t}-\alpha\hat{t}-\beta)\frac{1}{n}+\sum_{j<k}(\alpha t_j-\alpha\hat{t})\frac{1}{n} \quad \geq \quad (\hat{t}-\alpha\hat{t}-\beta)Q_k(t)+\sum_{j<k}(\alpha t_j-\alpha\hat{t})Q_j(t),$$

which is equivalent to

$$\underbrace{(\hat{t}-\alpha\hat{t}-\beta)}_{<0}\left(\frac{1}{n}-Q_k(t)\right)+\sum_{j<k}\underbrace{\alpha(t_j-\hat{t})}_{<0}\left(\frac{1}{n}-Q_j(t)\right) \quad \geq \quad 0,$$

where the underbraced inequalities follow from Assumption 1 and (34). The above inequality together with (35) implies $Q_j(t)=\frac{1}{n}$ for all $j\leq k$. Thus, $\sum_{j>k} Q_j(t) = 1-k/n$. From (31), $Q_{k+1}(t)=\cdots=Q_n(t)$, implying $Q_j(t)=1/n$ for all $j>k$. This completes the proof of (32). $\square$

Proof of Lemma 2. Using the definition (6),

$$yq_Y(y) \quad = \quad \sum_{j=0}^{n-1}\frac{(n-1)!y^{j+1}(1-y)^{n-(j+1)}}{(j+1)!(n-(j+1))!}p_{j+1} \quad = \quad \frac{1}{n}\sum_{j=0}^{n-1}B_y^n(j+1)p_{j+1}. \tag{36}$$

Taking derivatives on both sides yields

$$yq_Y'(y)+q_Y(y)=\frac{1}{n}\sum_{j=0}^{n-1}\frac{d}{dy}B_y^n(j+1)p_{j+1}.$$

Thus, using the following standard identity about the derivative of a Bernstein polynomial,

$$\begin{aligned}\frac{d}{dy}B_y^n(j) &= n\left(B_y^{n-1}(j-1)-B_y^{n-1}(j)\right) \text{ for } j=1,...,n-1, \\ \frac{d}{dy}B_y^n(j) &= nB_y^{n-1}(j-1) \text{ for } j=n\end{aligned} \tag{37}$$

we obtain

$$\begin{aligned}yq_Y'(y)+q_Y(y) &= \sum_{j=0}^{n-1}B_y^{n-1}(j)p_{j+1}-\sum_{j=0}^{n-2}B_y^{n-1}(j+1)p_{j+1} \\ &= \sum_{j=0}^{n-1}B_y^{n-1}(j)p_{j+1}-\sum_{j=0}^{n-1}B_y^{n-1}(j)p_j.\end{aligned}$$

Now (12) follows from the definitions (4) and (5). The proof of (13) is analogous. $\square$

Proof of Lemma 3. Consider any mechanism-equilibrium combination $(p_1',\ldots,p_{n-1}',\sigma')$. Let $y'=y(\sigma')$ denote the probability that an agent plays Y.

Any equilibrium with $y'=0$ or $y'=1$ yields the uniformly-random-assignment allocation. Thus, an equivalent mechanism-equilibrium combination in partition form is given by the uniformly-random-assignment rule together with any partition strategy y, that is, we can set $(p_1, \ldots, p_{n-1}, y) = (1/n, \ldots, 1/n, y)$.

Now suppose that $0 < y' < 1$. Then there exists a type $t_L < \hat{t} < t_H$ such that $U_Y(\sigma', \hat{t}) = U_N(\sigma', \hat{t})$. Thus, using (16),

$$U_Y(\sigma', t) - U_N(\sigma', t) = (q_Y(y') - q_N(y'))(t - \hat{t}) \quad \text{for all } t. \tag{38}$$

There are three cases, (i) $q_Y(y') > q_N(y')$, (ii) $q_Y(y') < q_N(y')$, and (iii) $q_Y(y') = q_N(y')$.

In case (i), (38) implies that the strategy σ' is the partition strategy y'. Thus, we can set $(p_1, \ldots, p_{n-1}, y) = (p'_1, \ldots, p'_{n-1}, y')$.

In case (ii), (38) implies that $\sigma'(t) = 1$ for all $t < \hat{t}$ and $\sigma'(t) = 0$ for all $t > \hat{t}$. We obtain an equivalent mechanism-equilibrium combination $(p_1, \ldots, p_{n-1}, \sigma)$ by setting $p_j = 1 - p'_{n-j}$ and $\sigma(t) = 1 - \sigma'(t)$. Thus σ has the partition form, showing that the desired conclusion holds with $y = 1 - y'$.

In case (iii), (11) implies $q_Y(y') = 1/n = q_N(y')$. In the case $u_Y = u_N$, the law of iterated expectations implies that $u_Y = u_N = \alpha\bar{t} + \beta$. Thus, equivalence to the uniformly-random assignment holds because the right-hand sides of (14) and (15) are equal to $(1 - \frac{1}{n})(\alpha\bar{t} + \beta) + \frac{1}{n}t$, which is the payoff from uniformly-random assignment.

Now suppose that $u_Y \neq u_N$. Using (16) and (38),

$$U_Y(\sigma', t) - U_N(\sigma', t) = (h_Y - h_N - \frac{1}{n})(u_Y - u_N) = 0 \quad \text{for all } t.$$

Thus, $h_Y - h_N - \frac{1}{n} = 0$. On the other hand, (9) implies that $y = yh_Y + (1 - y)h_N$. Solving the system of these two equations leads to the formulas in (8). Plugging these into (14) yields that

$$U_Y(\sigma, t) = (1 - \frac{1}{n})(\alpha\bar{t} + \beta) + \frac{1}{n}t.$$

Thus, the payoff from playing Y (and then also from playing N) is the same as in the uniformly-random-assignment rule.

□

Proof of Lemma 4. If $y = 0$ or $y = 1$, then the uniformly-random assignment obviously obtains. Thus, assume $0 < y < 1$.

If $q_Y(y) = q_N(y)$, then the conclusion follows from Lemma 3. Cases with $q_Y(y) < q_N(y)$ are analogous to cases with $q_Y(y) > q_N(y)$, with switched roles of the actions Y and N.

Thus, assume $q_Y(y) > q_N(y)$. Thus, y is a partition strategy. This implies $nyq_Y > y$ and $n(1 - y)q_N < (1 - y)$. Hence, (20) together with (17) implies

$$W(y) > yt_Y(y) + (1 - y)t_N(y) = \bar{t}, \tag{39}$$

where the last equality relies on the law of iterated expectations. Using (14) and (15), we see that, for any type t,

$$
\begin{aligned}
& yU_Y(y,t) + (1-y)U_N(y,t) \\
& = (y(h_Y - q_Y) + (1-y)h_N)\, u_Y + (y(1-h_Y) + (1-y)(1-h_N-q_N))\, u_N + \frac{1}{n}t
\end{aligned}
$$

Thus, using (9) and (10), we see that

$$
\begin{aligned}
& yU_Y(y,t) + (1-y)U_N(y,t) \\
= \;& (1-\frac{1}{n})(\alpha W(y) + \beta) + \frac{1}{n}t \\
> \;& (1-\frac{1}{n})(\alpha \bar{t} + \beta) + \frac{1}{n}t,
\end{aligned}
$$

which is the agent's payoff from a uniformly-random assignment. Therefore, also the equilibrium payoff $\max\{U_Y(y,t), U_N(y,t)\}$ is strictly larger than the payoff from a uniformly-random assignment. □

Proof of Remark 1. We do the maximization in two steps. First, we fix any y and solve for the optimal mechanism $(p_1, \ldots, p_{n-1})$ given y. Second, we maximize over y.

Using (11), we can alternatively write the objective W uniformly in terms of $q_Y(y)$, as

$$W(y) = nyq_Y(y)(t_Y(y) - t_N(y)) + t_N(y).$$

Therefore, the optimal mechanism maximizes $q_Y(y)$. Thus, from (6) the any-volunteer rule is the unique optimal mechanism if $0 < y < 1$. Moreover, the any-volunteer rule is *an* optimal mechanism if $y = 0$ or $y = 1$. Any binary-first-best y^* satisfies $0 < y^* < 1$ because otherwise the uniformly-random assignment would obtain. Thus, $W'(y^*) = 0$. □

Proof of Lemma 5. Using (20) and applying the product differentiation rule, we find

$$
\begin{aligned}
\frac{1}{n}W'(y) \;&=\; \frac{d}{dy}\left(yq_Y(y)t_Y(y) + (1-y)q_N(y)t_N(y)\right) \\
&= q_Y t_Y - q_N t_N + yq_Y'(y)t_Y + (1-y)q_N'(y)t_N + yq_Y t_Y'(y) + (1-y)q_N t_N'(y). \qquad (40)
\end{aligned}
$$

We first calculate the derivatives

$$t_Y'(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{y}\int_{\hat{t}(y)}^{t_H} t\mathrm{d}F(t)\right) = \frac{\hat{t}(y) - t_Y(y)}{y} \qquad (41)$$

and

$$t_N'(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{1-y}\int_{t_L}^{\hat{t}(y)} t\mathrm{d}F(t)\right) = \frac{t_N(y) - \hat{t}(y)}{1-y}. \qquad (42)$$

Plugging (12), (13), (41), and (42) into (40), and dropping the y-arguments on the right-hand side, we get

$$\begin{aligned}\alpha\frac{1}{n}W'(y) &= \alpha t_Y(h_Y - h_N - q_Y) + \alpha t_N(h_N - h_Y + q_N) + \alpha\hat{t}(q_Y - q_N)\\ &= u_Y(h_Y - h_N - q_Y) + u_N(h_N - h_Y + q_N) + (\alpha\hat{t} + \beta)(q_Y - q_N).\end{aligned}$$

Using (16) and the definition of $\Delta(y)$ in (18), we get that

$$\Delta(y) = u_Y(h_Y - h_N - q_Y) + u_N(h_N - h_Y + q_N) + \hat{t}(q_Y - q_N). \tag{43}$$

Combining the two equations, the claimed formula follows. □

Proof of Corollary 1. For any binary first best allocation y^*, it holds that $W'(y^*) = 0$ for W evaluated at the any-volunteer rule.

If $\beta - (1-\alpha)\hat{t}(y^*) = 0$, then Lemma 5 implies $\Delta(y^*) = 0$ for the any-volunteer rule, meaning that $y = y^*$ is a partition-equilibrium strategy of the any-volunteer rule.

Now suppose that $\beta - (1-\alpha)\hat{t}(y^*) \neq 0$ for all binary first best allocations y^*. Recall that $0 < y^* < 1$. For the any-volunteer rule, $q_Y(y^*) - q_N(y^*) > 0$ (see Proposition 4). Because $W'(y^*) = 0$, Lemma 5 implies $\Delta(y^*) \neq 0$, showing that y^* cannot be an equilibrium strategy of the any-volunteer rule.

□

Proof of Remark 2. There exists a sequence (y_n) such that $y_n \to 0$ and $(1-y_n)^n \to 0$. For example, taking $y_n = 1/\sqrt{n}$, it holds that $(1-y_n)^{\sqrt{n}} \to 1/e$ by definition of the Euler number e, and $(1/e)^{\sqrt{n}} \to 0$.

Given the any-volunteer rule, (4) and (5) imply that, for all y,

$$h_Y(y_n) = 1 \quad \text{and} \quad h_N(y_n) = 1 - (1-y_n)^{n-1}.$$

Thus, (9) implies that $ny_nq_Y(y_n) = 1 - (1-y_n)^n \to 1$. This together with $t_Y(y_n) \to t_H$ implies that $W(y_n) \to t_H$. Denoting by W_n^* the expected type of the selected agent in the binary first-best, obviously $W_n^* \geq W(y_n)$. Thus, $W_n^* \to t_H$.

For any n, let y_n^* denote a binary-first-best volunteering rate. Thus,

$$W_n^* = ny_n^*q_Y(y_n^*)(t_Y(y_n^*) - t_N(y_n^*)) + t_N(y_n^*).$$

We note that $y_n^* \to 0$ and $ny_n^*q_Y(y_n^*) \to 1$ because otherwise $\liminf_n t_Y(y_n^*) < t_H$ or $\liminf_n$ $ny_n^*q_Y(y_n^*) < 1$, implying $\liminf_n W_n^* < t_H$. □

Proof of Lemma 6. Consider any solution to the relaxed problem. Suppose first that $y = 1$. Then $q_Y(y) = 1/n$ by (6), implying that the value at the optimum of the relaxed problem equals $\bar{t}$. This value can also be reached within the feasible set of the planner's binary-second-best problem, by using the uniformly-random-assignment rule. Thus, both problems reach the same value at the optimum, as was to be shown.

Now consider cases in which $y < 1$. Suppose that $\Delta(y) > 0$. Applying Lemma 5, we see that

$$\frac{\alpha}{n}\frac{dW}{dy} > (q_Y(y) - q_N(y))((\alpha-1)\hat{t}(y) + \beta).$$

The right-hand side is ≥ 0 because of Assumption 1. This is a contradiction to optimality because none of the constraints on y is binding. □

Proof of Proposition 1. If $n = 2$, then we have nothing to prove because any binary mechanism is a threshold rule. Assume that $n \geq 3$.

Consider any solution $(p_1 \ldots, p_{n-1}, y)$. By Lemma 6, it also solves the relaxed problem.

From Lemma 4 and Proposition 5 below it follows that the value reached at a solution exceeds the value at a uniformly-random assignment. Thus, $0 < y < 1$ and $q_Y(y) > q_N(y)$ by Lemma 3.

Fixing y, the remaining relaxed maximization problem over $(p_1, \ldots, p_{n-1})$ is a linear problem. Hence the Lagrange conditions are necessary and sufficient, without any qualification. Let $\lambda \geq 0$ denote the Lagrange multiplier for the constraint $\Delta(y) \geq 0$. Due to $q_Y(y) > q_N(y)$, the Lagrange multiplier for the constraint $q_Y(y) - q_N(y) \geq 0$ equals 0. Thus, using the Lagrangian $L = W + \lambda\Delta$, for all $i = 1, \ldots, n-1$,

$$\begin{array}{l} \text{if } \frac{\partial L}{\partial p_i} > 0, \text{ then } p_i = 1, \\ \text{if } \frac{\partial L}{\partial p_i} < 0, \text{ then } p_i = 0. \end{array} \tag{44}$$

Using (20),

$$\frac{\partial W}{\partial p_i} = B_y^n(i)\,(t_Y - t_N) = \frac{B_y^{n-1}(i)}{n-i} n(1-y)\,(t_Y - t_N)\,.$$

Similarly, using the definition of Δ (see (18)),

$$\frac{\partial \Delta}{\partial p_i} = \frac{B_y^{n-1}(i)}{n-i}\left(\left(\frac{1-y}{y} - 1\right)\hat{t}(y) + \left(i\frac{1-y}{y} - (n-i)\right)(u_Y - u_N) - \frac{1-y}{y}u_Y + u_N\right).$$

Thus,

$$\frac{\partial L}{\partial p_i} = \underbrace{\frac{B_y^{n-1}(i)}{n-i}}_{>0}\left(i\lambda\underbrace{\left(\frac{1-y}{y} + 1\right)(u_Y - u_N)}_{>0} + [\text{terms independent of } i]\right). \tag{45}$$

Consider the case that $\lambda > 0$. If $\partial L/\partial p_i < 0$ for all i, then (44) implies that $(p_1 \ldots, p_{n-1}) = (0, \ldots, 0)$, a threshold rule. Otherwise let i^* be the smallest integer such that $\partial L/\partial p_i \geq 0$. Then (44) implies that $(p_1 \ldots, p_{n-1})$ is an i^*-threshold rule.

It remains to consider the case $\lambda = 0$. Then $\partial L/\partial p_i = \partial W/\partial p_i > 0$ for all i, implying $p_j = 1$ for all j, that is, the solution entails the any-volunteer rule. □

Proof of Lemma 7. For any mechanism-partition-equilibrium combination $(p_1, ..., p_{n-1}, y)$, the ex-post payoff of a volunteering type $t \geq \hat{t}(y)$ when there are i volunteers is given by $p_i u_Y(y) + (1-p_i)u_N(y) + \frac{p_i}{i}(t - u_Y(y))$. Similarly, the ex-post payoff of a non-volunteering type $t < \hat{t}(y)$ is given by $p_i u_Y(y) + (1-p_i)u_N(y) + \frac{1-p_i}{n-i}(t - u_N)$. A volunteering type $t \geq \hat{t}(y)$ receives more than the payoff from uniformly-random assignment, which is $\frac{t-u_Y}{n} + \frac{i}{n}u_Y + \frac{n-i}{n}u_N$, if and only if

$$i(p_i - \frac{i}{n})(u_Y - u_N) \quad \geq \quad (p_i - \frac{i}{n})(u_Y - t). \tag{46}$$

The analogous condition for a non-volunteering type $t < \hat{t}(y)$ is

$$(n-i)(p_i - \frac{i}{n})(u_Y - u_N) \quad \geq \quad (p_i - \frac{i}{n})(t - u_N). \tag{47}$$

Adding up these two conditions for $t = \hat{t}(y)$ yields

$$n(p_i - \frac{i}{n})(u_Y - u_N) \geq (p_i - \frac{i}{n})(u_Y - u_N).$$

This implies (22).

The arguments so far show that the constraints stated in the lemma are necessary. To see sufficiency, note that (22) together with (23) implies (46), and (47) follows because $u_Y \geq t$ by Assumption 1. ☐

Proof of Proposition 2. Because the value reached at a solution exceeds the value at a uniformly-random assignment, we have $0 < y < 1$ and $q_Y(y) > q_N(y)$ by Lemma 3.

We adapt the proof of Proposition 1, where the Lagrangian $L = W + \lambda\Delta$ is used. The analogue here is the Lagrangian

$$\mathcal{L} = L + \sum_{i=1}^{n-1} \mu_i (p_i - \frac{i}{n}) \left(i(u_Y - u_N) - (u_Y - \hat{t}) \right).$$

First note that in any solution to the planner's problem as augmented according to Lemma 7 it holds that $(n-1)(u_Y - u_N) \geq (u_Y - \hat{t})$ because otherwise the constraint (23) implies that $p_i = \frac{i}{n}$ for all i, yielding the uniformly-random assignment rule. Let $1 \leq \hat{\imath} \leq n-1$ be minimal with the property $\hat{\imath}(u_Y - u_N) \geq (u_Y - \hat{t})$. For all $i < \hat{\imath}$, the constraint (23) implies that $p_i = \frac{i}{n}$.

If $\lambda = 0$, then $\partial\mathcal{L}/\partial p_i \geq \partial L/\partial p_i = \partial W/\partial p_i > 0$ for all $i \geq \hat{\imath}$, implying $p_i = 1$ and we are done. Suppose that $\lambda > 0$.

Let $i^* \geq \hat{\imath}$ be minimal with the property $p_{i^*} > \frac{i^*}{n}$. Thus, $\partial\mathcal{L}/\partial p_{i^*} \geq 0$ by the Lagrangian conditions. It is sufficient to show that

$$\frac{\partial \mathcal{L}}{\partial p_{i^*}} \quad = \quad \frac{\partial L}{\partial p_{i^*}}. \tag{48}$$

Indeed, this implies $\partial L/\partial p_{i^*} \geq 0$ and hence, using (45), $\partial\mathcal{L}/\partial p_i \geq \partial L/\partial p_i > 0$ for all $i > i^*$, hence $p_i = 1$, and we are done.

The equation (48) is immediate if $\mu_{i^*} = 0$. If $\mu_{i^*} > 0$, then, by the Lagrangian conditions, the constraint (23) is binding, which implies $i^*(u_Y - u_N) - (u_Y - \hat{t}) = 0$ and hence again (48) holds. $\square$

Proof of Remark 3. Let y denote a partition equilibrium. The claim is trivial if $y = 0$ or $y = 1$, so assume that $0 < y < 1$. Then the equilibrium condition $\Delta(y) = 0$ holds, that is, an agent with marginal type $\hat{t}$ is indifferent between volunteering or not:

$$\sum_{j=i^*}^{n-1} B_y^{n-1}(j)\frac{1}{j+1}(\hat{t} - u_Y) + B_y^{n-1}(i^* - 1)\left((\frac{1}{i^*} - \frac{1}{n})(\hat{t} - u_Y) - \frac{n - i^*}{n}(u_N - u_Y)\right) = 0,$$

Because $\hat{t}(y) < t_Y < u_Y$ from Assumption 1, the sum $\sum_{j=i^*}^{n-1} \ldots$ above is strictly negative, implying that

$$(\frac{1}{i^*} - \frac{1}{n})(\hat{t}(y) - u_Y) - \frac{n - i^*}{n}(u_N - u_Y) \geq 0.$$

This inequality is equivalent to (23) at $i = i^*$, with $p_{i^*} = 1$, that is, the ex-post participation constraint at i^* volunteers is satisfied. From this the ex-post participation constraints (23) at $i > i^*$ volunteers are immediate. The ex-post participation constraints at $i < i^*$ volunteers are trivial because $p_i = i/n$. $\square$

Proof of Lemma 8. Before presenting the proof, we provide a roadmap. *Step 0* recalls the Poisson approximation of the binomial distribution.

Using the assumption on i^* made in the lemma, *Step 1* shows that there exists a sequence of strategies along which the expected number of volunteers does not vanish and the marginally volunteering type strictly prefers volunteering over not-volunteering. Lowering the marginal type until she becomes indifferent then yields a sequence of equilibria with non-vanishing expected numbers of volunteers z_n (this is *Step 2*).

In *Step 3* we show that z_n is bounded. Suppose otherwise, that is, $z_n \to \infty$ on some subsequence. That is, the expected number of volunteers tends to infinity. Given that only i^* volunteers are needed, the probability that the task is assigned to a non-volunteer falls to zero so fast along the subsequence that the probability tends to 0 even if it is first telescoped by z_n. On the other hand, an agent expects that she is selected with a probability approximately equal to $1/z_n$ if she volunteers. Therefore, after telescoping payoffs by z_n and defining the marginal type $\hat{t}_n = F^{-1}(1 - \hat{y}_n)$, with a large n type $\hat{t}_n$'s payoff difference between volunteering and non-volunteering is $\approx \hat{t}_n - u_Y(y_n)$, which is strictly negative by Assumption 1, contradicting equilibrium.

Step 4 considers any large-population limit point z^* of the sequence z_n (existence of a limit point follows from *Step 3*). We show that z^* satisfies the equation that is stated in the lemma. The left-hand-side is strictly decreasing by (24), showing that the limit point is unique. Elementary properties of the Poisson distribution are useful: the probability $\text{Pois}(z^*)(i^* - 1)$ differs by a factor of z^*/i^* from the probability that there are *exactly* i^* volunteers; this probability differs by a factor of z^*/i^* from the probability that there are *at least* i^* volunteers.

Step 5 establishes the formula for r^*.

Step 0. Consider any sequence of numbers (y_n), $y_n \in [0,1]$ and a number $z > 0$ such that $z_n \to z$, where we use the shortcuts $z_n = ny_n$. Then

$$\lim_n B_{y_n}^{n-1}(j) \quad = \quad e^{-z}\frac{z^j}{j!} \quad \text{for } j = 0, 1, \dots \tag{49}$$

and

$$\lim_n \sum_{j=i^*-1}^{n-1} B_{y_n}^{n-1}(j)\frac{1}{j+1} \quad = \quad e^{-z}\sum_{j=i^*-1}^{\infty}\frac{z^j}{(j+1)!}. \tag{50}$$

Formula (49) is the well-known Poisson limit theorem. To see (50), define $B_{y_n}^{n-1}(j) = 0$ for all $j > n-1$ and note that

$$\begin{aligned}\sum_{j=i^*-1}^{\infty} e^{-z}\frac{z^j}{(j+1)!} &\overset{(49)}{=} \sum_{j=i^*-1}^{\infty}\lim_n B_{y_n}^{n-1}(j)\frac{1}{j+1} \\ &= \lim_n \sum_{j=i^*-1}^{\infty} B_{y_n}^{n-1}(j)\frac{1}{j+1} = \lim_n \sum_{j=i^*-1}^{n-1} B_{y_n}^{n-1}(j)\frac{1}{j+1}.\end{aligned}$$

This completes *Step 0*.

Using the assumption on i^* made in the proposition, the argument of the function $\ln(\dots)$ below is larger than 1. Thus, we can fix a number $\underline{z}$ such that

$$0 < \underline{z} < \ln\left(\frac{\alpha(t_H - \bar{t})i^*}{\alpha t_H + \beta - t_H}\right). \tag{51}$$

We will use the functions h_Y, h_N, q_Y, and q_N, as applied to the threshold rule with parameter i^* and $p_{i^*} = 1$. (The proof extends to the corresponding threshold rule with uniformly-random default; the reason is that at any volunteering rate $y > 0$, the threshold i^* is reached for sure in the $n \to \infty$ limit.) Plugging in the definitions,

$$h_Y(y) - h_N(y) = B_y^{n-1}(i^* - 1), \tag{52}$$

$$q_Y(y) \quad = \quad \sum_{j=i^*-1}^{n-1} B_y^{n-1}(j)\frac{1}{j+1}, \tag{53}$$

and

$$q_N(y) \quad = \quad \sum_{j=0}^{i^*-1} B_y^{n-1}(j)\frac{1}{n-j}. \tag{54}$$

Using (16) and (18),

$$\Delta(y) = (q_Y - q_N)\hat{t}(y) + (h_Y - h_N - q_Y)u_Y - (h_Y - h_N - q_N)u_N. \tag{55}$$

Step 1. For all n, define $\underline{y}_n = \underline{z}/n$. Then $\Delta(\underline{y}_n) > 0$ for all sufficiently large n.

We take the limit $n \to \infty$ in (18) with $y = \underline{y}_n$.

Because $\underline{y}_n \to 0$, only the highest type volunteers in the large-population limit. Thus, $\hat{t}(\underline{y}_n) \to t_H$, $t_Y \to t_H$ and $t_N \to \bar{t}$.

Consider a threshold rule (it can be checked that all limits are identical for the corresponding threshold rule with uniformly-random default).

Applying (49) with $j = i^* - 1$ to (52),

$$\lim_n h_Y(\underline{y}_n) - h_N(\underline{y}_n) \quad = \quad e^{-\underline{z}}\frac{(\underline{z})^{i^*-1}}{(i^*-1)!}.$$

Applying (50) to (53),

$$\lim_n q_Y(\underline{y}_n) \quad = \quad e^{-\underline{z}} \sum_{j=i^*-1}^{\infty} \frac{\underline{z}^j}{(j+1)!}.$$

Using (54),

$$q_N(\underline{y}_n) \quad \leq \quad \frac{1}{n-i^*} \to 0 \quad \text{as } n \to \infty.$$

Thus, using (55) and cancelling terms,

$$\lim_n \Delta(\underline{y}_n) \quad = \quad e^{-\underline{z}}\frac{(\underline{z})^{i^*-1}}{(i^*-1)!}\alpha\underbrace{(t_H - \bar{t})}_{>0} + e^{-\underline{z}} \sum_{j=i^*-1}^{\infty} \frac{(\underline{z})^j}{(j+1)!}\underbrace{(t_H - (\alpha t_H + \beta))}_{<0 \text{ by Assumption 1}}, \tag{56}$$

where we have used that $u_Y - u_N \to \alpha(t_H - \bar{t})$ and $\hat{t} - u_Y \to t_H - (\alpha t_H + \beta)$.

Note that

$$\sum_{j=i^*-1}^{\infty} \frac{(\underline{z})^j}{(j+1)!} \leq \sum_{j=i^*-1}^{\infty} \frac{(\underline{z})^{i^*-1}(\underline{z})^{j-i^*+1}}{(i^*)!(j-i^*+1)!} = \frac{(\underline{z})^{i^*-1}}{(i^*)!}e^{\underline{z}}.$$

Thus, (56) implies

$$\begin{aligned}\lim_n \Delta(\underline{y}_n) &\geq \frac{(\underline{z})^{i^*-1}}{(i^*-1)!} e^{-\underline{z}} \alpha(t_H - \bar{t}) + \frac{(\underline{z})^{i^*-1}}{(i^*)!}(t_H - (\alpha t_H + \beta)) \\ &= \frac{(\underline{z})^{i^*-1}}{(i^*-1)!}\left(e^{-\underline{z}}\alpha(t_H - \bar{t}) - \frac{\alpha t_H + \beta - t_H}{i^*}\right) > 0,\end{aligned}$$

where the last inequality follows from (51). This completes *Step 1.*

Step 2. For all sufficiently large n, there exists a partition-equilibrium strategy y_n such that $z_n > \underline{z}$, where we define $z_n = ny_n$.

To see this, note first that $\Delta(1) < 0$ by Lemma 14. By *Step 1* and the Intermediate Value Theorem, there exists y_n such that $z_n > \underline{z}$ and $\Delta(y_n) = 0$. Let y_n be maximal with these properties.

For all threshold rules with $i^* \geq 2$, Proposition 5 implies that $q_Y(y_n) > q_N(y_n)$. For all threshold rules with uniformly-random default (which also includes the any-volunteer rule), Proposition 4 implies $q_Y(y_n) > q_N(y_n)$.

Thus, y_n is a partition-equilibrium strategy.

For the remaining steps, we consider any sequence of partition equilibria (y_n) such that $\liminf_n z_n > 0$.

Using (52) to (55) with any y,

$$\begin{aligned}\Delta(y) &= B_y^{n-1}(i^*-1)(u_Y(y) - u_N(y)) \\ &+ \sum_{j \geq i^*-1} B_y^{n-1}(j)\frac{1}{j+1}(\hat{t}(y) - u_Y(y)) - \sum_{j \leq i^*-1} B_y^{n-1}(j)\frac{1}{n-j}(\hat{t}(y) - u_N(y)).\end{aligned}$$

Multiplying the equilibrium condition $\Delta(y_n) = 0$ with z_n and using the shortcut $\hat{t}_n = \hat{t}(y_n)$, we obtain

$$\begin{aligned}0 &= z_n B_{y_n}^{n-1}(i^*-1)(u_Y(y_n) - u_N(y_n)) \\ &+ z_n \sum_{j \geq i^*-1} B_{y_n}^{n-1}(j)\frac{1}{j+1}(\hat{t}_n - u_Y(y_n)) - z_n \sum_{j \leq i^*-1} B_{y_n}^{n-1}(j)\frac{1}{n-j}(\hat{t}_n - u_N(y_n)).\end{aligned} \tag{57}$$

Step 3. The sequence (z_n) is bounded. (In particular, $y_n \to 0$.)

We do a proof by contradiction. Suppose that, along some subsequence, $z_n \to \infty$. Then, along this subsequence,

$$B_{y_n}^n(j) \to 0 \quad \text{for all } j = 0, 1, \ldots. \tag{58}$$

To see this, note that, due to elementary properties of the binomial distribution,

$$B_{y_n}^n(j) \leq n^j (y_n)^j (1-y_n)^{n-j} = (ny_n)^j (1-y_n)^{n-j},$$

implying

$$\ln\left(B^n_{y_n}(j)\right) \leq j\ln(ny_n) + (n-j)\ln(1-y_n).$$

Hence, using the elementary inequalities $\ln(1-y_n) \leq -y_n$ and $y_n \leq 1$,

$$\ln\left(B^n_{y_n}(j)\right) \leq j\ln(ny_n) - (n-j)y_n \leq j\ln(z_n) - z_n + j \to -\infty.$$

This implies (58).

From (58) it follows that

$$z_n B^{n-1}_{y_n}(j) \to 0 \quad \text{for all } j = 0, 1, \ldots. \tag{59}$$

To see this, note that, by elementary properties of the binomial distribution,

$$ny_n B^{n-1}_{y_n}(j) \quad = \quad (j+1)B^n_{y_n}(j+1). \tag{60}$$

From (59) it follows that

$$z_n \sum_{j=0}^{i^*-1} B^{n-1}_{y_n}(j) \to 0. \tag{61}$$

Now we consider the limit $n \to \infty$ in (57). By (59) and (61), the first and third terms vanish, and in the second term the range of the sum can be replaced by $\sum_{j\geq 0}$. Using (60),

$$\lim_n z_n \sum_{j=0}^{n-1} B^{n-1}_{y_n}(j)\frac{1}{j+1} = \lim_n \sum_{j=0}^{n-1} B^n_{y_n}(j+1) = 1 - \lim_n B^n_{y_n}(0) \overset{(58)}{=} 1.$$

Plugging this into (57) yields $\lim_n \hat{t}_n - u_Y(y_n) = 0$.

Because y_n is a partition strategy, the average volunteer type is larger than the marginal type, $t_Y(y_n) \geq \hat{t}_n$. Thus, using Assumption 1,

$$u_Y(y_n) - \hat{t}_n = \alpha t_Y(y_n) + \beta - \hat{t}_n \geq \alpha\hat{t}_n + \beta - \hat{t}_n \geq \min_{t\in[0,1]} \alpha t + \beta - t > 0,$$

which contradicts the 0 limit derived above. This completes *Step 3*.

Step 4. Consider any limit point z^* of (z_n). Then $h^{Pois(z^*)}(i^*) = \frac{\alpha t_H + \beta - t_H}{i^*\alpha(t_H - \bar{t})}$.

To see this, consider a subsequence $z_{n_k} \to z^*$ as $k \to \infty$. A computation analogous to that leading to (56) implies

$$\lim_k \Delta(y_{n_k}) \quad = \quad e^{-z^*}\frac{(z^*)^{i^*-1}}{(i^*-1)!}\alpha(t_H - \bar{t}) + e^{-z^*}\sum_{j=i^*-1}^{\infty}\frac{(z^*)^j}{(j+1)!}(t_H - (\alpha t_H + \beta)).$$

Applying the equilibrium condition $\Delta(y_{n_k}) = 0$,

$$0 \quad = \quad e^{-z^*}\frac{(z^*)^{i^*-1}}{(i^*-1)!}\alpha(t_H - \bar{t}) + e^{-z^*}\sum_{j=i^*-1}^{\infty}\frac{(z^*)^j}{(j+1)!}(t_H - (\alpha t_H + \beta)).$$

After multiplying by z^*/i^* and switching to the variable $j' = j+1$ in the sum,

$$0 \quad = \quad e^{-z^*}\frac{(z^*)^{i^*}}{i^*!}\alpha(t_H - \bar{t}) + \sum_{j'=i^*}^{\infty} e^{-z^*}\frac{(z^*)^{j'}}{j'!}\frac{t_H - (\alpha t_H + \beta)}{i^*}.$$

Thus,

$$0 \quad = \quad \mathrm{Pois}(z^*)(i^*)\alpha(t_H - \bar{t}) - \sum_{j'=i^*}^{\infty}\mathrm{Pois}(z^*)(j')\frac{\alpha t_H + \beta - t_H}{i^*}.$$

This implies the claimed formula, completing the proof of *Step 4*.

From (24) one sees that, for any i, the function $z \mapsto h^{\mathrm{Pois}(z)}(i)$ is strictly decreasing. Thus, the limit point z^* established in *Step 3* is unique, showing that the sequence (z_n) converges to z^*.

Step 5. The formula for r^*.

The probability that the task is assigned to a volunteer in the equilibrium with the partition strategy y_n is

$$r_n = \sum_{j=i^*}^{n} B^n_{y_n}(j) \overset{(60)}{=} \sum_{j=i^*}^{n}\frac{z_n}{j}B^{n-1}_{y_n}(j-1)$$

Thus, using *Step 0*,

$$\lim_n r_n = z^*\sum_{j=i^*}^{\infty}\frac{(z^*)^{j-1}}{j!}e^{-z^*} = \sum_{j=i^*}^{\infty}\frac{(z^*)^j}{j!}e^{-z^*}$$

This completes the proof of the lemma. □

Proof of Proposition 3. Using the shortcut $\kappa = (\alpha t_H + \beta - t_H)/(\alpha(t_H - \bar{t}))$, the equation in (25) can also be written as

$$i^*\mathrm{Pois}(z^*)(i^*) \quad = \quad \kappa\sum_{j=i^*}^{\infty}\mathrm{Pois}(z^*)(j)$$

or, using the definition of $\mathrm{Pois}(z^*)(i^*-1)$, and the definition of r^* (a function of i^* and z^*) in the statement of Lemma 8,

$$i^*\frac{e^{-z^*}(z^*)^{i^*}}{(i^*)!} \quad = \quad \kappa r^*. \tag{62}$$

We will use the following (Chernoff) bounds for tail probabilities as applied to a Poisson random variable with mean z:

$$\sum_{j=i}^{\infty} \mathrm{Pois}(z)(j) \quad \leq \quad \frac{e^{-z}(ez)^i}{i^i} \quad \text{for all } i \geq z, \tag{63}$$

$$\sum_{j=0}^{i} \mathrm{Pois}(z)(j) \quad \leq \quad \frac{e^{-z}(ez)^i}{i^i} \quad \text{for all } i \leq z. \tag{64}$$

To prove (63), let X denote a Poisson distributed random variable with mean z. Then

$$E[\left(\frac{i}{z}\right)^X] = \sum_{k=0}^{\infty} \left(\frac{i}{z}\right)^k \frac{z^k e^{-z}}{k!} = \sum_{k=0}^{\infty} \frac{i^k}{k!} e^{-z} = e^{i-z}.$$

Thus, (63) follows from the Markov inequality:

$$\Pr[X \geq i] = \Pr[\left(\frac{i}{z}\right)^X \geq \left(\frac{i}{z}\right)^i] \leq \frac{E[\left(\frac{i}{z}\right)^X]}{\left(\frac{i}{z}\right)^i} = e^{i-z}\left(\frac{z}{i}\right)^i.$$

The proof of (64) is analogous.

We begin by showing that

$$z^* \geq i^* \text{ for all sufficiently large } i^*. \tag{65}$$

Suppose that $z^* < i^*$. Using (63) and the definition of r^*,

$$r^* \quad \leq \quad \frac{e^{-z^*}(ez^*)^{i^*}}{{i^*}^{i^*}}.$$

Using (62), we can substitute r^* and obtain

$$i^* \frac{e^{-z^*}(z^*)^{i^*}}{(i^*)!} \quad \leq \quad \kappa \frac{e^{-z^*}(ez^*)^{i^*}}{{i^*}^{i^*}}.$$

After cancelling terms,

$$\frac{{i^*}^{i^*+1}}{(i^*)! e^{i^*}} \quad \leq \quad \kappa.$$

By Stirling's formula, the left-hand side tends to infinity as $i^* \to \infty$, yielding a contradiction. This shows (65).

In particular, $\sqrt{i^*-1}/z^* \to 0$ as $i^* \to \infty$. Because the right-hand side of (62) is bounded by κ, it also follows that

$$\frac{e^{-z^*}(z^*)^{i^*-1}\sqrt{i^*-1}}{(i^*-1)!} \quad \to \quad 0.$$

By Stirling's formula,

$$\frac{e^{i^*-1-z^*}(z^*)^{i^*-1}}{(i^*-1)^{(i^*-1)}} \quad \to \quad 0.$$

Thus, using (64) with $i = i^* - 1$,

$$1 - r^* \leq \frac{e^{i^*-1-z^*}(z^*)^{i^*-1}}{(i^*-1)^{(i^*-1)}} \to 0,$$

implying $\lim_{i^* \to \infty} r^* = 1$. $\square$

Lemma 14. *Consider any mechanism* $(p_1, \ldots, p_{n-1})$ *such that* $p_{n-1} > 1 - 1/n$. *Then* $q_Y(1) > q_N(1)$ *and* $\Delta(1) < 0$, *implying that there is no equilibrium with* $y = 1$.

Proof. The strategy in which $y = 1$ leads to the following values of the relevant functions:

$$t_Y(1) = \bar{t}, \quad h_Y(1) = 1, \quad h_N(1) = p_{n-1}, \quad q_Y(1) = \frac{1}{n}, \quad q_N(1) = 1 - p_{n-1}. \tag{66}$$

In particular, $q_Y(1) > q_N(1)$. Because $\hat{t}(1) = t_L$, plugging the expressions from (66) into the definition of $\Delta(y)$ yields that

$$\Delta(1) = U_Y(1, t_L) - U_N(1, t_L) = \underbrace{(\frac{1}{n} - 1 + p_{n-1})}_{>0} \underbrace{(t_L - \alpha\bar{t} - \beta)}_{<0},$$

where the second underbraced inequality follows from Assumption 1. $\square$

Proof of Proposition 4. Recall that $1 \leq i^* \leq n-1$, $p_i = i/n$ for all $i < i^*$, and $p_i = 1$ for all $i > i^*$.

Note that $i^* + 1 \leq n$ implies $1/n \leq 1/(i^*+1)$, thus $(n - i^*)/n \leq (n-i^*)/(i^*+1)$, or

$$1 - i^*/n \leq (n - i^*)/(i^* + 1). \tag{67}$$

Hence,

$$1 - p_{i^*} \quad < \quad \frac{n - i^*}{i^* + 1}. \tag{68}$$

Using the definitions (6) and (7), for any y,

$$q_Y(y) - q_N(y)$$
$$= B_y^{n-1}(i^*-1)\underbrace{\left(\frac{p_{i^*}}{i^*} - \frac{1}{n}\right)}_{>0} + B_y^{n-1}(i^*)\underbrace{\left(\frac{1}{i^*+1} - \frac{1-p_{i^*}}{n-i^*}\right)}_{>0 \text{ by } (68)} + \sum_{j=i^*+1}^{n-1} B_y^{n-1}(j)\underbrace{\left(\frac{1}{j+1} - 0\right)}_{>0}.$$

If $y > 0$, then at least one of the binomial probabilities above is strictly positive, implying $q_Y(y) > q_N(y)$.

Thus, the right-hand side in (16) is strictly increasing in t, showing that any equilibrium is a partition equilibrium with some cut-off type $\hat{t}$ such that all types $t < \hat{t}$ play N and all types $t > \hat{t}$ play Y.

To get a better understanding of equilibrium, we need an explicit expression for $\Delta(y)$. Using (4), (5), and (6),

$$\begin{aligned} h_Y - h_N - q_Y &= \sum_{j=0}^{i^*-2} B_y^{n-1}(j)\frac{1}{n} + B_y^{n-1}(i^*-1)(p_{i^*} - \frac{i^*-1}{n}) + B_y^{n-1}(i^*)(1-p_{i^*}) \\ &\quad - \sum_{j=0}^{i^*-2} B_y^{n-1}(j)\frac{1}{n} - B_y^{n-1}(i^*-1)\frac{p_{i^*}}{i^*} - B_y^{n-1}(i^*)\frac{1}{i^*+1} - \sum_{i^*+1}^{n-1} B_y^{n-1}(j)\frac{1}{j+1} \\ &= B_y^{n-1}(i^*-1)\cdot(i^*-1)\left(\frac{p_{i^*}}{i^*} - \frac{1}{n}\right) + B_y^{n-1}(i^*)\left(\frac{i^*}{i^*+1} - p_{i^*}\right) \\ &\quad - \sum_{i^*+1}^{n-1} B_y^{n-1}(j)\frac{1}{j+1}. \end{aligned}$$

Similarly, using (4), (5), and (7),

$$h_Y - h_N - q_N = B_y^{n-1}(i^*-1)\cdot i^*\left(\frac{p_{i^*}}{i^*} - \frac{1}{n}\right) + B_y^{n-1}(i^*)\cdot(1-p_{i^*})\left(1 - \frac{1}{n-i^*}\right).$$

Let us first consider cases in which $t_H + (i^*-1)(\alpha t_H + \beta) > i^*(\alpha\bar{t} + \beta)$.

Consider $y \approx 0$. Then $B_y^{n-1}(i^*-1)$ is much larger then $B_y^{n-1}(j)$ for all $j \geq i^*$, implying

$$\begin{aligned} \frac{q_Y - q_N}{B_y^{n-1}(i^*-1)} &\approx \frac{p_{i^*}}{i^*} - \frac{1}{n}, \\ \frac{h_Y - h_N - q_Y}{B_y^{n-1}(i^*-1)} &\approx (i^*-1)\left(\frac{p_{i^*}}{i^*} - \frac{1}{n}\right), \\ \frac{h_Y - h_N - q_N}{B_y^{n-1}(i^*-1)} &\approx i^*\left(\frac{p_{i^*}}{i^*} - \frac{1}{n}\right). \end{aligned}$$

Thus,

$$\frac{\Delta(y)}{B_y^{n-1}(i^*-1)} \approx \left(\frac{p_{i^*}}{i^*} - \frac{1}{n}\right)\left(\hat{t} + (i^*-1)u_Y - i^* u_N\right).$$

From $y \approx 0$ it follows that $\hat{t} \approx t_H$, $u_Y \approx \alpha t_H + \beta$, and $u_N \approx \alpha\bar{t} + \beta$. Thus, in the case considered, $\hat{t} + (i^*-1)u_Y - i^* u_N > 0$ if y is close to 0, implying that $\Delta(y) > 0$ if y is close to 0. From Lemma 14, we know that $\Delta(1) < 0$. By continuity of Δ, there exists y such that $\Delta(y) = 0$, yielding the desired equilibrium.

Now consider cases in which

$$t_H + (i^* - 1)(\alpha t_H + \beta) \leq i^*(\alpha \bar{t} + \beta). \tag{69}$$

By Lemma 14, it is sufficient to show that $\Delta(y) < 0$ for all $0 < y < 1$. Moreover, because $\Delta(y)$ is linear in p_{i^*}, it is sufficient to consider the extreme cases $p_{i^*} = 1$ and $p_{i^*} \approx i^*/n$ or, for that matter, $p_{i^*} = i^*/n$.

In fact, it is sufficient to consider the case $p_{i^*} = 1$ for any i^*. The case $p_{i^*} = i^*/n$ leads to the rule $(p_1, \ldots, p_{n-1}) = (1/n, \ldots, i^*/n, 1, \ldots, 1)$, which is already covered by the first case, with an incremented definition of i^*. The only case not covered by this then is the case $i^* = n - 1$, $p_{i^*} = i^*/n$; this is the uniformly-random assignment rule, where $\Delta(y) = 0$ for all y; by linearity of $\Delta(y)$ in p_{n-1}, we conclude that $\Delta(y) < 0$ for all other threshold rules with uniformly random default.

Assume $p_{i^*} = 1$. Note that, using Assumption 1, $\hat{t} < \alpha\hat{t} + \beta < u_Y$. Thus, if after plugging the expressions for $q_Y - q_N$ and $h_Y - h_N - q_Y$ that were obtained above into (86) we drop the sums $\sum_{i^*+1}^{n-1}$, then we obtain a strict upper bound for $\Delta(y)$, that is,

$$\begin{aligned} \Delta(y) \quad < \quad & B_y^{n-1}(i^* - 1) \\ & \cdot \left(\left(\frac{1}{i^*} - \frac{1}{n} \right) (t_Y + (i^* - 1)u_Y - i^* u_N) + \frac{y}{1-y} \frac{n - i^*}{i^*} \frac{1}{i^* + 1} (t_Y - u_Y) \right). \end{aligned}$$

where we have also used the inequality $\hat{t} < t_Y$, have plugged in the above expression for $h_Y - h_N - q_N$, and have used that

$$B_y^{n-1}(i^*) = B_y^{n-1}(i^* - 1) \frac{y}{1-y} \frac{n - i^*}{i^*}.$$

Now using (67) we obtain

$$\frac{\Delta(y)}{B_y^{n-1}(i^* - 1) \cdot (\frac{1}{i^*} - \frac{1}{n})} \quad < \quad t_Y + (i^* - 1)u_Y - i^* u_N + \frac{y}{1-y}(t_Y - u_Y).$$

The assumption (69) implies

$$t_Y + (i^* - 1)(\alpha t_Y + \beta) \leq i^*(\alpha \bar{t} + \beta),$$

which can also be written as

$$t_Y - u_Y \leq i^* \alpha(\bar{t} - t_Y).$$

Thus,

$$\frac{\Delta(y)}{B_y^{n-1}(i^* - 1) \cdot (\frac{1}{i^*} - \frac{1}{n})} \quad < \quad i^* \alpha(\bar{t} - t_N) + \frac{y}{1-y} i^* \alpha(\bar{t} - t_Y)).$$

Hence,

$$\frac{\Delta(y)(1-y)}{B_y^{n-1}(i^*-1)\cdot(\frac{1}{i^*}-\frac{1}{n})i^*\alpha} \quad < \quad (1-y)(\bar{t}-t_N)+y(\bar{t}-t_Y) \;=\; 0,$$

where the equation follows from the law of iterated expectations. Thus, $\Delta(y) < 0$. □

Lemma 15. *If $n = 2$, then the threshold rule with $i^* = 1$ and $p_{i^*} = 1/2$ satisfies $h_Y(y) - h_N(y) = 1/2$ for all $y \in [0, 1]$. For any other threshold rule if $n = 2$, and for any threshold rule if $n \geq 3$, there exists $y^{*m} \in [0, 1]$ such that, for all $y \in (0, 1)$,*

$$(h_Y - h_N)'(y) > 0 \text{ if } y < y^{*m}, \text{ and } (h_Y - h_N)'(y) < 0 \text{ if } y > y^{*m}. \tag{70}$$

Proof of Lemma 15. Plugging in the definitions,

$$h_Y(y) - h_N(y) = B_y^{n-1}(i^*)(1-p_{i^*}) + B_y^{n-1}(i^*-1)p_{i^*}. \tag{71}$$

Assume first that $n = 2$. Then $i^* = 1$, and (71) implies that $h_Y - h_N = y(1-p_{i^*}) + (1-y)p_{i^*}$. Thus, $(h_Y - h_N)'(y) = 1 - 2p_{i^*}$. If $p_{i^*} = 1/2$, then the difference $h_Y - h_N$ is constant and equal to 1/2. If $p_{i^*} < 1/2$, then $(h_Y - h_N)'(y) > 0$ for all $y \in [0, 1]$, so that we can set $y^{*m} = 1$. If $p_{i^*} > 1/2$, then $(h_Y - h_N)'(y) < 0$ for all $y \in [0, 1]$, so that we can set $y^{*m} = 0$.

Now assume that $n \geq 3$. Suppose that $i^* = 1$. Using (71), it is straightforward to verify that

$$(h_Y - h_N)'(y) \quad = \quad (1-y)^{n-3}(n-1)l(y),$$

where we use the auxiliary function

$$l(y) \quad = \quad 1 - 2p_{i^*} - y(n(1-p_{i^*}) - 1),$$

which is linear in y. If $p_{i^*} \geq 1 - 1/n$, then $l(0) = 1 - 2p_{i^*} < 0$ and $l(1) = (n-2)(p_{i^*} - 1) \leq 0$, implying that $l(y) < 0$ for all $y \in [0, 1)$. Thus, (70) holds with $y^{*m} = 0$. If $p_{i^*} < 1 - 1/n$, then $l(y)$ is strictly decreasing in y, implying (70).

The case $i^* = n - 1$ is treated analogously to the case $i^* = 1$.

Suppose that $1 < i^* < n - 1$. Using (71), it is straightforward to verify that

$$(h_Y - h_N)'(y) \quad = \quad \frac{(n-1)!y^{i^*-2}(1-y)^{n-2-i^*}}{(i^*)!(n-i^*)!}l(y),$$

where we use the auxiliary function

$$\begin{aligned} l(y) \quad = \quad & y(1-y)(1-2p_{i^*})i^*(n-i^*) \\ & - y^2(1-p_{i^*})(n-i^*)(n-i^*-1) \;+\; (1-y)^2 p_{i^*} i^*(i^*-1). \end{aligned}$$

Note that $l(0) = (i^* - 1)i^* p_{i^*} > 0$ if $p_{i^*} > 0$, and $l'(0) = i^*(n - i^*) > 0$ if $p_{i^*} = 0$, implying that

$$l(y) > 0 \quad \text{for all } y > 0 \text{ that are sufficiently close to 0.}$$

Similarly, $l(1) = -(n-i^*)(n-i^*-1)(1-p_{i^*}) < 0$ if $p_{i^*} < 1$, and $l'(1) = i^*(n-i^*) > 0$ if $p_{i^*} = 1$, implying that

$$l(y) < 0 \quad \text{for all } y < 1 \text{ that are sufficiently close to 1.}$$

Thus, by the mean-value theorem, $l(y^{*m}) = 0$ for some $y^{*m} \in (0, 1)$. Moreover, y^{*m} is unique because l is quadratic in y. Hence, for all $y \in (0, 1)$,

$$l(y) > 0 \text{ if } y < y^{*m}, \text{ and } l(y) < 0 \text{ if } y > y^{*m},$$

showing (70). □

Proof of Lemma 9. We proceed in two steps. The first step is to show that for any non-extreme threshold rule,

$$h_Y(y) - h_N(y) - q_Y(y) > 0 \text{ for all } y > 0 \text{ sufficiently close to 0,} \tag{72}$$

$$\text{and } h_Y(y) - h_N(y) - q_Y(y) < 0 \text{ for all } y < 1 \text{ sufficiently close to 1.} \tag{73}$$

This implies that the functions q_Y and $h_Y - h_N$ are non-identical polynomials that, by the intermediate-value theorem, intersect at least once.

The second step is to show that given (72) and (73), the functions q_Y and $h_Y - h_N$ intersect at most once.

For the first step, let $(p_1, ..., p_{n-1})$ be any non-extreme threshold rule. Let i^* be minimal such that $p_{i^*} > 0$. Then

$$q_Y(y) \quad = \quad \sum_{j=i^*}^{n-1} B_y^{n-1}(j) \frac{1}{j+1} + B_y^{n-1}(i^* - 1) \frac{p_{i^*}}{i^*}. \tag{74}$$

Using this and (71), we see that

$$h_Y(y) - h_N(y) = \binom{n-1}{i^*-1} y^{i^*-1} p_{i^*} + \mathcal{O}(y^{i^*}), \quad q_Y(y) = \binom{n-1}{i^*-1} y^{i^*-1} \frac{p_{i^*}}{i^*} + \mathcal{O}(y^{i^*}),$$

Thus, (72) holds if $i^* \geq 2$. If $i^* = 1$, then

$$\begin{aligned} h_Y(y) - h_N(y) &= p_1 + (n-1)y(1-p_1) - (n-1)yp_1 + \mathcal{O}(y^2), \\ q_Y(y) &= p_1 + (n-1)y\frac{1}{2} - (n-1)yp_1 + \mathcal{O}(y^2). \end{aligned}$$

Thus, (72) follows from $1 - p_1 > 1 - 1/n > 1/2$. Using (77) and the assumption $p_{n-1} > 1 - \frac{1}{n}$,

$$h_Y(1) - h_N(1) - q_Y(1) = 1 - p_{n-1} - \frac{1}{n} < 0,$$

implying (73).

For the second step, note that the functions q_Y and $h_Y - h_N$, being polynomials, can have only a finite number of intersection points. We have to show that they intersect at most once.

From (12), the intersection points are the critical points of q_Y. We can also calculate the second derivative for all $y \in (0,1)$:

$$q''_Y(y) = \frac{(h_Y(y) - h_N(y))' - 2q'_Y(y)}{y}.$$

Thus, at any critical point y of q_Y, the number $q''_Y(y)$ has the same sign as $(h_Y(y) - h_N(y))'$. Hence, using Lemma 15, any critical point $y \in (0, y^{*m})$ is a local maximum, and any critical point $y \in (y^{*m}, 1)$ is a local minimum. Taking into account (72) and (73), it becomes clear that q_Y must be decreasing at first and intersect $h_Y - h_N$ at its minimum to the right of y^{*m}. □

Proof of Proposition 5. Given that the conditions of Lemma 9 are satisfied, there exists y^{m1} as stated there. From (12) and Lemma 9, the function q_Y is strictly increasing on $[0, y^{m1}]$ and is strictly decreasing on $[y^{m1}, 1]$. Note that $q_Y(0) = p_1 < 1/n$ and, using (77), $q_Y(1) = 1/n$. Thus, there exists a unique $\check{y} < y^{m1}$ such that $q_Y(\check{y}) = 1/n$. Moreover,

$$q_Y(y) > 1/n \quad \text{for all} \quad y \in (\check{y}, 1). \tag{75}$$

Now Lemma 9 implies that

$$h_Y(\check{y}) - h_N(\check{y}) - q_Y(\check{y}) \quad > \quad 0. \tag{76}$$

Moreover, $q_N(\check{y}) = 1/n = q_Y(\check{y})$. Applying the definition of Δ at $y = \check{y}$,

$$\Delta(\check{y}) \quad = \quad \underbrace{(h_Y(\check{y}) - h_N(\check{y}) - q_Y(\check{y}))}_{>0 \text{ by (76)}} \underbrace{(u_Y - u_N)}_{>0 \text{ by (17)}}.$$

On the other hand, $\Delta(1) < 0$ by Lemma 14. Thus, $\hat{y}$ as defined in the statement of the proposition satisfies $\check{y} < \hat{y} < 1$. From (75), $q_Y(\hat{y}) > 1/n$, implying $q_Y(\hat{y}) > q_N(\hat{y})$ by (11). The better-off claim follows from Lemma 4. □

Proofs for the case of a pleasant task

Proof of Lemma 1'. The only difference to the proof of the corresponding lemma in the unpleasant-task case is that we consider an ordered type profile $t_1 \geq \cdots \geq t_n$. This leads to the inequality

$$\underbrace{(\hat{t} - \alpha\hat{t} - \beta)}_{>0} \underbrace{(\frac{1}{n} - Q_k(t))}_{\leq 0} + \sum_{j<k} \underbrace{\alpha(t_j - \hat{t})}_{>0} \underbrace{(\frac{1}{n} - Q_j(t))}_{\leq 0} \quad \geq \quad 0,$$

where one of the underbraced inequalities follows from Assumption 1'. The above inequality then implies

$$Q_j(t) \quad = \quad \frac{1}{n} \quad \text{for all } j \leq k,$$

and the remaining argument is again as in the proof of of the corresponding lemma in the unpleasant-taks case. □

Proof of Proposition 5'. Note first that for any mechanism $(p_1, \ldots, p_{n-1})$ such that $p_1 > 1/n$, it holds that $\Delta(0) > 0$, implying that there is no equilibrium with $y = 0$. This is shown by calculating the relevant functions for $y = 0$:

$$t_N(0) = \bar{t}, \quad h_Y(0) = p_1, \quad h_N(0) = 0, \quad q_Y(0) = p_1, \quad q_N(0) = \frac{1}{n}. \tag{77}$$

Because $\hat{t}(0) = t_H$, plugging the expressions from (77) into the definition of $\Delta(y)$ yields that

$$\Delta(0) = \underbrace{(p_1 - \frac{1}{n})}_{>0} \underbrace{(t_H - \alpha\bar{t} - \beta)}_{>0},$$

where the second underbraced inequality follows from Assumption 1'.

Now let $(p_1, ..., p_{n-1})$ be any non-extreme max-threshold rule. The proof considers the shape of the functions q_Y and q_N. Because of (12), we can write

$$\Delta(y) = (q_Y - q_N)(\hat{t} - u_N) + yq'_Y(u_Y - u_N).$$

Moreover, (11) implies that $q_Y(y) > q_N(y)$ is equivalent to $q_Y(y) > \frac{1}{n}$ and that $yq'_Y + q_Y = -(1-y)q'_N + q_N$. The latter generalizes (e.g. by induction) to

$$yq_Y^{(k)}(y) + kq_Y^{(k-1)}(y) = -(1-y)q_N^{(k)}(y) + kq_N^{(k-1)}(y) \text{ for any } k \geq 1, \tag{78}$$

where $q_Y^{(k)}$ denotes the k-th derivative of q_Y. From (77), we see that $q_Y(0) > q_N(0)$ at $y = 0$. Calculations show that

$$q_Y(1) = \frac{1}{n} < 1 - p_{n-1} = q_N(1).$$

This implies that the functions q_Y and q_N are non-identical polynomials that, by the intermediate value theorem, intersect at least once. There must therefore be a point $0 < y^1 < 1$ with $q_Y(y^1) = q_N(y^1)$ and $q_Y(y) > q_N(y)$ for all $0 \leq y < y^1$. There are three possiblities for the sign of $q'_Y(y^1)$:

Case 1: $q'_Y(y^1) < 0$

This implies that $\Delta(y^1) = y^1 q'_Y(y^1) < 0$. By the intermediate-value theorem, there exists a $y^* < y^1$ with $\Delta(y^*) = 0$ and Δ decreasing at $y = y^*$. By definition of y^1, it must be that

$q_Y(y^*) > q_N(y^*)$, which means that the condition for strict improvement over random assignment in Proposition 4 is satisfied at y^*. This equilibrium is dynamically stable.

Case 2: $q'_Y(y^1) > 0$

This implies $q'_N(y^1) = \frac{-y^1 q'_Y(y^1)}{1-y^1} < 0$. That q_Y is strictly increasing and q_Y is strictly decreasing at y^1 is a contradiction to the definition of y^1.

Case 3: $q'_Y(y^1) = 0$

This implies $q'_N(y^1) = 0$ and also $\Delta(y^1) = 0$. Since $q''_Y(y^1)$ and $q''_N(y^1)$ have opposite signs, there are two possibilities.

The first possibility is that the two functions both have saddle points at y^1, with q_Y strictly decreasing and q_N strictly increasing. This implies that $q_Y^{(k)}(y_1) < 0$ for some derivative of order $k \geq 3$ with k odd and all derivatives of lower order are equal to zero. As a consequence, also $q'_N(y^1) = ... = q_N^{(k-1)}(y^1) = 0$ and $q_N^{(k)}(y^1) > 0$. Consider now Δ and its derivatives at $y = y^1$. One obvious implication is $\Delta(y^1) = 0$, but also Δ', which is a sum of terms that all include as factors either $q_Y - q_N$, q'_y, q'_N, or q''_Y, must be equal to zero at $y = y^1$. Continuing this logic, if all derivatives up to the k-th derivative of q_Y and q_N are zero at $y = y^1$, then also $\Delta^{(k-1)}(y^1) = y^1 q_Y^{(k)}(u_Y(y^1) - u_N(y^1)) < 0$. Hence, at this saddle-point of q_Y and q_N, the function Δ must have a local maximum. But if $\Delta(y^1) = 0$ is a local maximum, then Δ must be negative to the left of y^1, such that the intermediate value theorem yields existence of a $0 < y^* < y^1$ with $\Delta(y^*) = 0$ and Δ decreasing at $y = y^*$.

The second possibility is that at y^1, q_Y has a local minimum and q_N has a local maximum. If this is the case, then to the right of y^1 it holds that $q_Y > q_N$ and $q'_Y > 0$, implying $\Delta(y) > 0$. There must then exist y^2 with $q_Y(y^2) = q_N(y^2)$ and $q_Y > q_N$ on the interval (y_1, y_2). The case distinction can then be repeated for y^2, and since $q_Y - q_N$ is a polynomial, there can only be finitely many such points.

□

Proof of Lemma 6'. Consider any solution to the relaxed problem. Suppose first that $y = 0$. Then $q_N(y) = 1/n$ by (11), implying that the value at the optimum of the relaxed problem equals $\bar{t}$. This value can also be reached within the feasible set of the planner's binary-second-best problem, by using the uniformly-random-assignment rule. Thus, both problems reach the same value at the optimum, as was to be shown.

Now consider cases in which $y > 0$. Suppose that $\Delta(y) < 0$. Applying Lemma 5, we see that

$$\frac{\alpha}{n}\frac{dW}{dy} < (q_Y(y) - q_N(y))((\alpha - 1)\hat{t}(y) + \beta).$$

The right-hand side is ≤ 0 because of Assumption $1'$. This is a contradiction to optimality because none of the constraints on y is binding. □

Proof of Proposition 1'. If $n = 2$, then we have nothing to prove because any binary mechanism is a maximum-threshold rule. Assume that $n \geq 3$.

Consider any solution $(p_1 \ldots, p_{n-1}, y)$. By Lemma 6', it also solves the relaxed problem.

From Lemma 4 and Proposition 5' below it follows that the value reached at a solution exceeds the value at a uniformly-random assignment. Thus, $0 < y < 1$ and $q_Y(y) > q_N(y)$ by Lemma 3.

Fixing y, the remaining relaxed maximization problem over $(p_1, \ldots, p_{n-1})$ is a linear problem. Hence the Lagrange conditions are necessary and sufficient, without any qualification. Let $\lambda \leq 0$ denote the Lagrange multiplier for the constraint $\Delta(y) \leq 0$. Due to $q_Y(y) > q_N(y)$, the Lagrange multiplier for the constraint $q_Y(y) - q_N(y) \geq 0$ equals 0. Thus, using the Lagrangian $L = W + \lambda\Delta$, for all $i = 1, \ldots, n-1$,

$$\begin{array}{l} \text{if } \frac{\partial L}{\partial p_i} > 0, \text{ then } p_i = 1, \\ \text{if } \frac{\partial L}{\partial p_i} < 0, \text{ then } p_i = 0. \end{array} \tag{79}$$

The derivative of the Lagrangian can be obtained as in the proof of Proposition 1':

$$\frac{\partial L}{\partial p_i} = \underbrace{\frac{B_y^{n-1}(i)}{n-i}}_{>0} \left(i\lambda \underbrace{\left(\frac{1-y}{y} + 1\right)(u_Y - u_N)}_{>0} + [\text{terms independent of } i] \right). \tag{80}$$

Consider the case that $\lambda < 0$. If $\partial L/\partial p_i > 0$ for all i, then (79) implies that $(p_1 \ldots, p_{n-1}) = (1, \ldots, 1)$, a maximum-threshold rule. Otherwise let i^* be the smallest integer such that $\partial L/\partial p_i \leq 0$. Then (79) implies that $(p_1 \ldots, p_{n-1})$ is a maximum-threshold rule.

It remains to consider the case $\lambda = 0$. Then

$$\frac{\partial L}{\partial p_i} = \frac{\partial W}{\partial p_i} > 0$$

for all i, implying $(p_1 \ldots, p_{n-1}) = (1, \ldots, 1)$, a maximum-threshold rule. □

Proof of Lemma 7'. For any mechanism-partition-equilibrium combination $(p_1, ..., p_{n-1}, y)$, one can show as in the proof of the corresponding Lemma 7 that a volunteering type $t \geq \hat{t}(y)$ receives more than the payoff from uniformly-random assignment if and only if

$$i(p_i - \frac{i}{n})(u_Y - u_N) \quad \geq \quad (p_i - \frac{i}{n})(u_Y - t). \tag{81}$$

The analogous condition for a non-volunteering type $t < \hat{t}(y)$ is

$$(n-i)(p_i - \frac{i}{n})(u_Y - u_N) \quad \geq \quad (p_i - \frac{i}{n})(t - u_N). \tag{82}$$

Adding up these two conditions for $t = \hat{t}(y)$ yields

$$n(p_i - \frac{i}{n})(u_Y - u_N) \geq (p_i - \frac{i}{n})(u_Y - u_N).$$

This implies (27).

The arguments so far show that the constraints stated in the lemma are necessary. To see sufficiency, note that (27) together with (28) implies (82), and (81) follows because $u_N \leq t$ by Assumption 1'. $\square$

Proof of Proposition 2'. Because the value reached at a solution exceeds the value at a uniformly-random assignment, we have $0 < y < 1$ and $q_Y(y) > q_N(y)$ by Lemma 3.

We adapt the proof of Proposition 1', where the Lagrangian $L = W + \lambda\Delta$ is used. The analogue here is the Lagrangian

$$\mathcal{L} = L + \sum_{i=1}^{n-1} \mu_i (p_i - \frac{i}{n}) \left((n-i)(u_Y - u_N) - (\hat{t} - u_N) \right).$$

First note that in any solution to the planner's problem as augmented according to Lemma 7' it holds that $(n-1)(u_Y - u_N) \geq (\hat{t} - u_N)$ because otherwise the constraint (28) implies that $p_i = \frac{i}{n}$ for all i, yielding the uniformly-random assignment rule. Let $1 \leq \hat{i} \leq n-1$ be maximal with the property $(n - \hat{i})(u_Y - u_N) \geq (\hat{t} - u_N)$. For all $i > \hat{i}$, the constraint (28) implies that $p_i = \frac{i}{n}$.

If $\lambda = 0$, then $\frac{\partial \mathcal{L}}{\partial p_i} \geq \frac{\partial L}{\partial p_i} = \frac{\partial W}{\partial p_i} > 0$ for all $i \geq \hat{i}$, implying $p_i = 1$ and we are done. Suppose that $\lambda < 0$.

Let $i^* \leq \hat{i}$ be maximal with the property $p_{i^*} > \frac{i^*}{n}$. Thus, $\frac{\partial \mathcal{L}}{\partial p_{i^*}} \geq 0$ by the Lagrangian conditions. It is sufficient to show that

$$\frac{\partial \mathcal{L}}{\partial p_{i^*}} = \frac{\partial L}{\partial p_{i^*}}. \tag{83}$$

Indeed, this implies $\frac{\partial L}{\partial p_{i^*}} \geq 0$ and hence, using (80), $\frac{\partial \mathcal{L}}{\partial p_i} \geq \frac{\partial L}{\partial p_i} > 0$ for all $i < i^*$, hence $p_i = 1$, and we are done.

The equation (83) is immediate if $\mu_{i^*} = 0$. If $\mu_{i^*} > 0$, Then, by the Lagrangian conditions, the constraint (28) is binding, which implies $(n - i^*)(u_Y - u_N) - (\hat{t} - u_N) = 0$ and hence again (83) holds. $\square$

Proof of Remark 3'. Let y denote a partition equilibrium. The claim is trivial if $y = 0$ or $y = 1$, so assume that $0 < y < 1$. It is sufficient to show that

$$(n - i^*)(u_Y - u_N) \geq \hat{t}(y) - u_N. \tag{84}$$

Indeed, bringing everything to the left-hand side and multiplying by $1 - \frac{i^*}{n}$ shows that the inequality is equivalent to (28) at $i = i^*$, with $p_{i^*} = 1$, that is, the ex-post participation constraint at i^* volunteers is satisfied. From this the ex-post participation constraints (28) at $i < i^*$ volunteers are immediate. The ex-post participation constraints at $i > i^*$ volunteers are trivial because $p_i = i/n$.

The equilibrium condition $\Delta(y) = 0$ can be written as

$$B_y^{n-1}(0)\frac{n-1}{n}(\hat{t} - u_N) + \sum_{j=1}^{i^*-1} B_y^{n-1}(j)\frac{(\hat{t} - u_Y)}{j+1} + B_y^{n-1}(i^*)\left(\frac{1}{n}(\hat{t} - u_N) - \frac{n-i^*}{n}(u_Y - u_N)\right) = 0,$$

or

$$\left(B_y^{n-1}(0)\frac{n-1}{n} + \sum_{j=1}^{i^*-1} B_y^{n-1}(j)\frac{1}{j+1} + B_y^{n-1}(i^*)\frac{1}{n}\right)(\hat{t} - u_N)$$
$$= \left(\sum_{j=1}^{i^*-1} B_y^{n-1}(j)\frac{1}{j+1} + B_y^{n-1}(i^*)\frac{n-i^*}{n}\right)(u_Y - u_N).$$

Suppose that (84) fails. Then the above inequality implies

$$\left(B_y^{n-1}(0)\frac{n-1}{n} + \sum_{j=1}^{i^*-1} B_y^{n-1}(j)\frac{1}{j+1} + B_y^{n-1}(i^*)\frac{1}{n}\right)(n - i^*)$$
$$< \sum_{j=1}^{i^*-1} B_y^{n-1}(j)\frac{1}{j+1} + B_y^{n-1}(i^*)\frac{n-i^*}{n}.$$

Equivalently,

$$B_y^{n-1}(0)\frac{n-1}{n}(n-i^*) + \sum_{j=1}^{i^*-1} B_y^{n-1}(j)\frac{1}{j+1}(n-i^*-1) \;<\; 0,$$

which is impossible because $i^* \leq n-1$. □

Proof of Proposition 4'. Note that $p_{i^*} > i^*/n$, implying

$$1 - p_{i^*} \quad < \quad \frac{n-i^*}{n}. \tag{85}$$

Using the definitions (6) and (7), for any y,

$$q_Y(y) - q_N(y)$$
$$= B_y^{n-1}(0)(1 - \frac{1}{n}) + \sum_{j=1}^{i^*-2} B_y^{n-1}(j)\frac{1}{j+1} + B_y^{n-1}(i^*-1)\frac{p_{i^*}}{i^*} + B_y^{n-1}(i^*)\underbrace{(\frac{1}{n} - \frac{1-p_{i^*}}{n-i^*})}_{>0 \text{ by } (85)}.$$

If $y < 1$, then at least one of the binomial probabilities above is strictly positive, implying $q_Y(y) > q_N(y)$.

Thus, the right-hand side in (16) is strictly increasing in t, showing that any equilibrium is a partition equilibrium with some cut-off type $\hat{t}$ such that all types $t < \hat{t}$ play N and all types $t > \hat{t}$

play Y. According to Lemma 4, every $0 < y < 1$ with $\Delta(y) = 0$ is an equilibrium that yields a strict improvement over uniformly-random assignment. Recall that

$$\Delta(y) \quad = \quad (q_Y - q_N)(\hat{t} - u_N) + (h_Y - h_N - q_Y)(u_Y - u_N). \tag{86}$$

Since $\Delta(0) = (p_1 - \frac{1}{n})(t_H - \bar{u}) > 0$, everybody volunteering is the unique equilibrium if and only if $\Delta(y) > 0$ for all $0 < y < 1$. Conversely, if we find a point where $\Delta < 0$, we can conclude that a better equilibrium exists. Note further that $\Delta(1) = (p_{n-1} - \frac{n-1}{n})(t_L - \bar{u})$, implying that for any non-extreme max-threshold rule, or if $t_L > \bar{u}$, there is an equilibrium in which everybody volunteers. To see whether this equilibrium is dynamically stable, we need to know how Δ behaves close to 1.

Using (4), (5), and (6),

$$h_Y - h_N - q_Y \quad = \quad -\sum_{j=1}^{i^*-2} B_y^{n-1}(j)\frac{1}{j+1} - B_y^{n-1}(i^*-1)(1 - \frac{i^*-1}{i^*}p_{i^*}) + B_y^{n-1}(i^*)(\frac{i^*}{n} - p_{i^*}).$$

Consider $y \approx 1$. Then $B_y^{n-1}(i^*)$ is much larger then $B_y^{n-1}(j)$ for all $j < i^*$, implying

$$\begin{aligned} \frac{q_Y - q_N}{B_y^{n-1}(i^*)} &\approx \frac{1}{n} - \frac{1 - p_{i^*}}{n - i^*}, \\ \frac{h_Y - h_N - q_Y}{B_y^{n-1}(i^*)} &\approx \frac{i^*}{n} - p_{i^*} = -(n - i^*)\left(\frac{1}{n} - \frac{1 - p_{i^*}}{n - i^*}\right). \end{aligned}$$

Thus,

$$\frac{\Delta(y)}{B_y^{n-1}(i^*)} \quad \approx \quad \underbrace{(\frac{1}{n} - \frac{1 - p_{i^*}}{n - i^*})}_{>0 \text{ by } (85)} \left(\hat{t} - u_N - (n - i^*)(u_Y - u_N)\right).$$

From $y \approx 1$ it follows that $\hat{t} \approx t_L$, $u_Y \approx \alpha\bar{t} + \beta$, and $u_N \approx \alpha t_L + \beta$. Let us first consider cases in which $t_L - u_L < (n - i^*)\alpha(\bar{t} - t_L)$. In this case, $\hat{t} - u_N + (n - i^*)(u_Y - u_N) < 0$ if y is close to 1, implying that $\Delta(y) < 0$ if y is close to 1. By continuity of Δ, there exists y such that $\Delta(y) = 0$, yielding the desired equilibrium. Moreover, this equilibrium is dynamically stable while the all-volunteer equilibrium, should it exist, is not.

Now consider cases in which $t_L - u_L > (n - i^*)\alpha(\bar{t} - t_L)$. This implies that $\Delta(y) \geq 0$ and $\Delta(y) > 0$ for $y < 1$ close to 1. Hence, everybody volunteering is a dynamically stable equilibrium.

Finally, consider the case that $\hat{t} - u_N \geq (n - i^*)\alpha(t_Y - t_N)$ for all $y \in [0, 1]$. Note that for $y = 1$, this coincides with $t_L - u_L \geq (n - i^*)\alpha(\bar{t} - t_L)$. For some settings, the latter condition already implies the former. If for example F is uniform and $\alpha \leq 2$, then $t_Y - t_N = \bar{t} - t_L$ for all y and $\hat{t} - u_N - (t_L - u_L) = (1 - \alpha/2)(\hat{t} - t_L) \geq 0$.

In order to show that everybody volunteering is the unique equilibrium, it is sufficient to show that $\Delta(y) > 0$ for all $0 < y < 1$. Moreover, because $\Delta(y)$ is linear in p_{i^*}, it follows as in the proof of Proposition 4 that it is sufficient to consider the extreme case $p_{i^*} = 1$. With $p_{i^*} = 1$,

$$\Delta(y) = B_y^{n-1}(0)\frac{n-1}{n}(\hat{t}-u_N) + \sum_{j=1}^{i^*-1} B_y^{n-1}(j)\frac{(\hat{t}-u_Y)}{j+1} + B_y^{n-1}(i^*)\frac{1}{n}(\hat{t}-u_N-(n-i^*)(u_Y-u_N)).$$

Since $\hat{t} - u_N > 0$, the first term is positive. Our assumption that $\hat{t} - u_N \geq (n - i^*)(u_Y - u_N)$ implies that the last term is nonnegative. It also implies that $\hat{t} - u_Y \geq (n - i^* - 1)(u_Y - u_N)$ and therefore $\hat{t} - u_Y > 0$. Thus, $\Delta(y) > 0$ for all $0 < y < 1$.

□

Proof of Lemma 8'. The proof follows the roadmap of the proof of the corresponding Lemma 8. *Step 0.* Consider any sequence of numbers (y_n), $y_n \in [0, 1]$ and a number $z > 0$ such that $z_n \to z$, where we use the shortcuts $z_n = ny_n$. Then we can apply the Poisson limit theorem to get the formula

$$\lim_n B_{y_n}^{n-1}(j) \quad = \quad e^{-z}\frac{z^j}{j!} \quad \text{for } j = 0, 1, \ldots. \tag{87}$$

Step 1. For all n, define $\overline{y}_n = i^*/n$. Step 1 is to show that $\Delta(\overline{y}_n) < 0$ for all sufficiently large n.

We will use the functions h_Y, h_N, q_Y, and q_N, as applied to the max-threshold rule with parameter i^* and $p_{i^*} = 1$. (The proof extends to the corresponding max-threshold rule with uniformly-random default; the reason is that at any volunteering rate $y > 0$, the number of volunteers remains below the threshold i^* for sure in the $n \to \infty$ limit.) Plugging in the definitions,

$$h_Y(y) - h_N(y) = B_y^{n-1}(0) + B_y^{n-1}(n-1) - B_y^{n-1}(i^*), \tag{88}$$

$$q_Y(y) \quad = \quad \sum_{j=0}^{i^*-1} B_y^{n-1}(j)\frac{1}{j+1} + B_y^{n-1}(n-1)\frac{1}{n}, \tag{89}$$

and

$$q_N(y) \quad = \quad B_y^{n-1}(0)\frac{1}{n} + \sum_{j=i^*+1}^{n-1} B_y^{n-1}(j)\frac{1}{n-j}. \tag{90}$$

Using (16) and (18),

$$\Delta(y) = (q_Y - q_N)\hat{t}(y) + (h_Y - h_N - q_Y)u_Y - (h_Y - h_N - q_N)u_N. \tag{91}$$

We take the limit $n \to \infty$ in (91) with $y = \overline{y}_n$.

Because $\overline{y}_n \to 0$, no type volunteers in the large-population limit. Thus, $\hat{t}(\overline{y}_n) \to t_H$, $t_Y \to t_H$ and $t_N \to \overline{t}$.

Consider a max-threshold rule (it can be checked that all limits are identical for the corresponding max-threshold rule with uniformly-random default).

Applying (87) alternatively with $j = 0$, $j = i^* - 1$, and $j = n - 1$, to (88),

$$\lim_n h_Y(\overline{y}_n) - h_N(\overline{y}_n) \quad = \quad e^{-i^*} - e^{-i^*}\frac{(i^*)^{i^*}}{i^*!}.$$

Applying (87) to (89),

$$\lim_n q_Y(\overline{y}_n) \quad = \quad \sum_{j=0}^{i^*-1} e^{-i^*}\frac{(i^*)^j}{j!}\frac{1}{j+1}.$$

By elementary properties of the binomial distribution, $B_y^{n-1}(j)/(n-j) = (y/(1-y))B_y^{n-1}(j-1)/j$. Therefore, (90) implies

$$q_N(\overline{y}_n) = B_{\overline{y}_n}^{n-1}(0)\frac{1}{n} + \frac{\overline{y}_n}{1-\overline{y}_n}\sum_{j=i^*+1}^{n-1} B_{\overline{y}_n}^{n-1}(j-1)\frac{1}{j} \leq \frac{1}{n} + \frac{1}{i^*+1}\frac{\overline{y}_n}{1-\overline{y}_n} \to 0 \text{ as } n \to \infty.$$

Thus, using (91) and cancelling terms,

$$\lim_n \Delta(\overline{y}_n) \quad = \quad \left(e^{-i^*} - e^{-i^*}\frac{(i^*)^{i^*}}{i^*!}\right)\alpha\underbrace{(t_H - \overline{t})}_{>0} + e^{-i^*}\sum_{j=0}^{i^*-1}\frac{(i^*)^j}{(j+1)!}\underbrace{(t_H - (\alpha t_H + \beta))}_{>0 \text{ by Assumption 1'}}, \quad (92)$$

where we have used that $u_Y - u_N \to \alpha(t_H - \overline{t})$ and $\hat{t} - u_Y \to t_H - (\alpha t_H + \beta)$.

Note that

$$e^{-i^*}\sum_{j=0}^{i^*-1}\frac{(i^*)^j}{(j+1)!} = e^{-i^*}\frac{1}{i^*}\sum_{j=0}^{i^*-1}\frac{(i^*)^{j+1}}{(j+1)!} \leq \frac{1}{i^*}.$$

Thus, (92) together with (29) implies

$$\lim_n \Delta(\overline{y}_n) \quad < \quad 0.$$

This completes *Step 1.*

Step 2. Step 2 is to show that for all sufficiently large n, there exists a partition-equilibrium strategy y_n such that $z_n < i^*$, where we define $z_n = ny_n$.

To see this, note first that $\Delta(0) > 0$ (see the proof of 5'). By *Step 1* and the intermediate value theorem, there exists y_n such that $z_n < i^*$ and $\Delta(y_n) = 0$. Let y_n be maximal with these properties.

For all max-threshold rules with $i^* \leq n-2$, Proposition 5' implies that $q_Y(y_n) > q_N(y_n)$. For all max-threshold rules with uniformly-random default (which also includes the any-volunteer rule, that is, $i^* = n-1$), Proposition 4' implies $q_Y(y_n) > q_N(y_n)$.

Thus, y_n is a partition-equilibrium strategy.

Step 3. Step 3 is to show that for any limit point z^* of (z_n), the formula in (30) holds.

In this pleasant-task situation, the expected number of volunteers remains below the threshold i^*, but converges to infinity if i^* converges to infinity.

To see this, consider a subsequence $z_{n_k} \to z^*$ as $k \to \infty$. A computation analogous to that leading to (92) implies

$$\lim_k \Delta(y_{n_k}) = \left(e^{-z^*} - e^{-z^*}\frac{(z^*)^{i^*}}{i^*!}\right)\alpha(t_H - \bar{t}) + e^{-z^*}\sum_{j=0}^{i^*-1}\frac{z^{*j}}{(j+1)!}(t_H - (\alpha t_H + \beta)),$$

Applying the equilibrium condition $\Delta(y_{n_k}) = 0$,

$$0 = \left(e^{-z^*} - e^{-z^*}\frac{(z^*)^{i^*}}{i^*!}\right)\alpha(t_H - \bar{t}) + e^{-z^*}\sum_{j=0}^{i^*-1}\frac{z^{*j}}{(j+1)!}(t_H - (\alpha t_H + \beta)).$$

After switching to the variable $j' = j + 1$ in the sum,

$$0 = \left(e^{-z^*} - e^{-z^*}\frac{(z^*)^{i^*}}{i^*!}\right)\alpha(t_H - \bar{t}) + e^{-z^*}\frac{1}{z^*}\sum_{j'=1}^{i^*}\frac{z^{*j'}}{(j')!}(t_H - (\alpha t_H + \beta)).$$

Thus,

$$0 = -\left(\mathrm{Pois}(z^*)(i^*) - \mathrm{Pois}(z^*)(0)\right)\alpha(t_H - \bar{t}) + \frac{1}{z^*}\sum_{j'=1}^{i^*}\mathrm{Pois}(z^*)(j')(t_H - (\alpha t_H + \beta)),$$

which is equivalent to the formula given in (30). This completes the proof of *Step 3.*

Step 4. The formula for r^*.

The probability that the task is assigned to a volunteer in the equilibrium with the partition strategy y_n is

$$r_n = \sum_{j=1}^{i^*} B^n_{y_n}(j) + B^n_{y_n}(n).$$

Thus, using *Step 0*,

$$\lim_n r_n = \sum_{j=1}^{i^*}\frac{(z^*)^j}{j!}e^{-z^*}$$

This completes the proof of the lemma. □

Proof of Proposition 3'. For each i^*, let $z^* \leq i^*$ denote a solution to (30). That is, throughout the proof z^* is considered as a function of i^*.

As in the main part, we will use a (Chernoff) bound for tail probabilities as applied to a

Poisson random variable with mean z:

$$\sum_{j=i}^{\infty} \text{Pois}(z)(j) \quad \leq \quad \frac{e^{-z}(ez)^i}{i^i} \quad \text{for all } i \geq z. \tag{93}$$

Applying this with $i = i^* + 1$ and using the definition of r^* in the statement of Lemma 8', it is sufficient to show that

$$1 - r^* = \frac{e^{-z^*}(ez^*)^{i^*+1}}{(i^*+1)^{(i^*+1)}} + e^{-z^*} \to 0 \quad \text{as } i^* \to \infty. \tag{94}$$

Using the shortcut $\kappa = (t_H - (\alpha t_H + \beta))/(\alpha(t_H - \bar{t}))$, the equation in (30) can also be written as

$$\frac{\text{Pois}(z^*)(i^*) - \text{Pois}(z^*)(0)}{\frac{1}{z^*}\sum_{j=1}^{i^*} \text{Pois}(z^*)(j)} = \kappa.$$

or, using the definition of $\text{Pois}(z^*)(i^*)$ and the definition of r^*,

$$z^* e^{-z^*}\left(\frac{(z^*)^{i^*}}{(i^*)!} - 1\right) \quad = \quad \kappa r^*. \tag{95}$$

Note that

$$z^* \to \infty. \tag{96}$$

Indeed, if there exists a finite limit point $\hat{z}$, then the left-hand side in (95) tends to $-\hat{z}e^{-\hat{z}} < 0$, contradicting (95).

Now (95) implies that

$$\limsup_{i^* \to \infty} e^{-z^*}\frac{(z^*)^{i^*+1}}{(i^*)!} \quad \leq \quad \kappa$$

Thus,

$$\limsup_{i^* \to \infty} \frac{e^{-z^*}(ez^*)^{i^*+1}}{(i^*+1)^{(i^*+1)}} \leq \kappa \limsup_{i^* \to \infty} \frac{e^{i^*+1}(i^*)!}{i^*(i^*)^{i^*}} = 0, \tag{97}$$

where the last equation follows from Stirling's formula. This proves (94). □

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