Renegotiation-Proof Relational Contracts

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Abstract

We study infinitely repeated two player games with perfect monitoring and assume that each period consists of two stages: one in which the players simultaneously choose an action and one in which they can transfer money to each other. In the first part of the paper, we derive simple conditions that allow a constructive characterization of all Pareto-optimal subgame perfect payoffs for all discount factors. In the second part, we examine different concepts of renegotiation-proofness and extend the characterization to renegotiation-proof payoffs.

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1 Introduction

Relational contracts are self-enforcing informal agreements that arise in many long-term relationships, often in response to obstacles to write endogenously enforceable contracts. Examples include the non-contractible aspects of employment relations, illegal cartel agreements, or buyer-seller relations in which complete formal contracts are too costly to write. Agreements between countries also often have the nature of a relational contract, when there is no institution that is able or willing to enforce compliance with the agreed terms. In these examples, monetary transfers play a role in the relationships, be it in form of prices, bonuses, or other compensation schemes, and could thus also be used to sustain the relational contract. Moreover, the relational contracts are drafted and negotiated, and meetings continue to take place as the relationship unfolds. In this paper, we analyze relational contracts under these circumstances: with renegotiation and the possibility to make monetary transfers.

As an illustration how side-payments can be used in a relational contract, consider the case of collusive agreements. Cartels sometimes use compensation schemes to make sure that each firm in the cartel stays with the target (see Harrington (2006) for details[1]). A cartel member that violates the agreement is required to buy a certain quantity from a competitor, or to transfer a valuable customer to a competitor. Such compensation schemes seem more robust to renegotiation than threatening with an immediate price war after a violation of the agreement. Price wars are costly for all firms, and therefore cartel members will be tempted to agree to ignore the violation. In contrast, if a deviating firm must pay a fine, competitors gain from the punishment and have therefore no incentive to renegotiate the agreement. However, to induce a firm to pay the compensation there must be the threat of a real punishment in case no payment is made, i.e., a punishment that does not require the voluntary participation of the punished firm. Renegotiation can then again play a role.

The present paper investigates these issues and provides a characterization of feasible payoffs given arbitrary discount factors. We study infinitely repeated two player games with perfect information in which players can make monetary transfers before they play a simultaneous move stage game. We translate Abreu's (1988) optimal penal codes to this set-up and show that every Pareto optimal subgame perfect payoff can be achieved using a class of simple strategy profiles, which we call *stationary contracts*. In a stationary contract, the same action profile is played in every period. A player who deviates from an action is required to pay a fine to the other player,

[1] For a list of cartels in which such compensation schemes have been used see the introduction of Harrington and Skrzypacz (2007), who offer a theoretical analysis of collusion with imperfect monitoring.

and after payment equilibrium play is resumed. If a player does not make a required payment, a punishment action profile is played once and then a lower fine is imposed on the player. Afterwards play continues as on the equilibrium path. Monetary transfers are useful not only because of the possibility to punish with fines, but also because any feasible distribution of the surplus can be achieved by incentive compatible up-front payments. In addition, equilibrium path payments in later periods can be used to balance incentive constraints between the two players.[2]

In the second part of the paper, we analyze different concepts of renegotiation-proofness and show that one can typically restrict the analysis to the simple class of stationary contracts to find payoffs that survive these renegotiation-proofness refinements. All considered concepts share the idea that renegotiation is deterred if players cannot achieve a Pareto improvement by renegotiating to an alternative continuation equilibrium chosen from an appropriate set.[3] Since a period consists of two stages, a crucial question is at what times renegotiation is possible. In the existing literature, different assumptions have (often implicitly) been made. For example, Fong and Surti (2009) assume in their study of repeated prisoner's dilemma games with side-payments that renegotiation is possible before both the payment and the play stage, while Levin (2003) assumes in his study of repeated principal-agent relationships that renegotiation is only possible before the payment stage.

Levin observes that the possibility of renegotiation before the payment stage does not alter the Pareto frontier of implementable payoffs. This observation easily extends to our set-up in which both players can take actions. The reason is that punishment at the payment stage takes the form of the deviator paying a fine to the other player immediately followed by a return to equilibrium play. Hence, in a Pareto efficient stationary contract, all continuation equilibria that start at the payment stage achieve the highest feasible joint continuation payoff. This means that if renegotiation is allowed only before payments can be made, the threat of inefficient continuation play (which is necessary to induce payment of the fine) is never subject to renegotiation.

For the main part of the analysis we also allow renegotiation before the play stage. Even having fixed the timing of renegotiation, there exist several concepts of renegotiation-proofness.

[2]In Goldlücke and Kranz (2012), we show how the subgame perfection results naturally extend to n players. In this companion paper, we analyze games with imperfect public monitoring, and addresses the case of perfect monitoring as a special case. It also contains a detailed discussion of how monetary transfers facilitate the computation of equilibrium payoffs and a comparison with computation procedures if no transfers are allowed.

[3]That Pareto improvements are necessary for successful renegotiation can be motivated by the idea that the original continuation equilibrium provides the fall back option (or disagreement point) if renegotiations should fail. See Miller and Watson (2012) for an alternative approach to model renegotiation and disagreement points in a repeated games with transfers.

We first adapt strong perfection (see Rubinstein, 1980) to our setting. A subgame perfect equilibrium is strong perfect if all its continuation payoffs lie on the Pareto frontier of subgame perfect continuation payoffs. In general, the set of strong perfect equilibrium payoffs is a subset of the Pareto frontier of subgame perfect payoffs, but it may well be empty. We show that every strong perfect payoff can be achieved by a stationary contract and derive simple conditions that allow to check for strong perfection. These conditions are used to show that in a simple principal agent game strong perfect stationary contracts always exist, while in other games they fail to exist.

We then analyze the concepts of weak renegotiation-proofness (WRP) and strong renegotiation-proofness (SRP) introduced by Farrell and Maskin (1989). An equilibrium is WRP if none of its continuation equilibria Pareto-dominate each other. This captures the idea that a necessary condition for renegotiation-proofness is that players never want to renegotiate to an alternative continuation equilibrium of the original contract.[4] Strong renegotiation-proofness requires that all continuation equilibria lie on the Pareto frontier of weakly renegotiation-proof payoffs. We show that if the discount factor is not below $\frac{1}{2}$, every Pareto-optimal WRP and every SRP payoff can be achieved by a stationary contract. The set of SRP equilibria may be empty for intermediate discount factors, but we provide simple sufficient conditions to check for existence. For discount factors below $\frac{1}{2}$, stationary contracts cannot always be used; instead the implementation of Pareto-optimal WRP payoffs can sometimes require alternation between different action profiles or money burning on the equilibrium path, as we illustrate for a prisoner's dilemma game.

Our analysis is most closely related to the work of Baliga and Evans (2000), who study asymptotic behavior of SRP equilibria in a setting in which payments and actions are chosen simultaneously. They establish that the set of SRP payoffs converges to the Pareto frontier of individually rational stage game payoffs when players become infinitely patient. Since under simultaneous choice of actions and payments inefficient action profiles are subject to renegotiation, their set-up is more closely related to our analysis where renegotiation before play stage is possible than to the case where only renegotiation before payment stages is considered. Our approach differs because we allow arbitrary discount factors and a sequential choice of actions and payments. While for discount factors close to one it does not matter whether payments and actions are made simultaneously or sequentially, for lower discount factor a sequential timing is more powerful: withholding a payment following a defection can then already be used as a punishment.

Fong and Surti (2009) study infinitely repeated prisoner's dilemma games with side

[4]Weak renegotiation-proofness is the terminology of Farrell and Maskin (1989). Bernheim and Ray (1989) introduce an essentially identical concept and call it *internal consistency*.

payments using the same timing as we do. They look at intermediate discount factors that can differ between players and derive sufficient conditions under which Pareto-optimal subgame perfect payoffs can be implemented as a WRP equilibrium. The framework with an impatient and a patient player makes their results on renegotiation-proofness complicated and difficult to interpret. In contrast, in our framework with a common discount factor, finding the sets of strong perfect, WRP or SRP payoffs is often not difficult, and the representation via stationary contracts helps to understand the technical conditions that describe these sets.

When Fong and Surti (2009) determine the set of subgame perfect payoffs in the prisoner's dilemma, they find that a restriction to stationary equilibrium paths is possible in this particular game, and conjecture that this should hold more generally. This is where the present paper ties in and shows that even with potentially complex punishments, stationary contracts are sufficient to describe the Pareto frontier of attainable payoffs. The result that for general two player stage games optimal subgame perfect equilibria can always have a stationary equilibrium path is also in line with previous work on relational contracts, including models of principal agent relationships (Levin, 2003 and Rayo, 2007), business partnerships (Baker et al., 2002, Blonski and Spagnolo, 2007, or Doornik, 2006) or collusion (Miklos-Thal, 2011). None of these more applied articles, however, extends the analysis to optimal punishment paths in general games. The contribution of the first part of our paper hence lies in the characterization of optimal penal codes in games with perfect monitoring and side payments.

The paper is organized as follows. In Section 2 we describe the model and introduce stationary contracts. Section 3 first establishes that all Pareto-optimal subgame perfect payoffs can be implemented by a stationary contract. We then explain a simple heuristic to characterize the Pareto frontier of subgame perfect payoffs for all discount factors. Section 4 first briefly establishes that renegotiation only before the payment stage does not restrict the set of Pareto-optimal subgame perfect payoffs. We then allow for renegotiation at both stages and use stationary contracts to derive simple conditions that characterize strong perfect payoffs. In a similar fashion, Section 5 characterizes weakly and strongly renegotiation-proof payoffs and exemplifies the derived conditions. All proofs are relegated to the appendix.

2 Model and Stationary Contracts

2.1 The game

We consider an infinitely repeated two-player game with perfect monitoring and common discount factor $\delta \in (0,1)$. Players are indexed by $i, j \in \{1,2\}$ and we use the convention that $j \neq i$ if both i and j appear in an expression. Every period t comprises two substages, without discounting between the substages: a *payment stage,* in which both players choose a nonnegative monetary transfer to the other player, and a *play stage,* in which the players play a simultaneous move game.

The stage game of the play stage is given by a continuous payoff function $g : A_1 \times A_2 \to \mathbb{R} \times \mathbb{R}$, where the set A_i is the compact action space of player i. We denote action profiles of this stage game by $a = (a_1, a_2)$ and the set of all action profiles by $A = A_1 \times A_2$. The joint payoff from an action profile action a is denoted by $G(a) = g_1(a) + g_2(a)$. The best reply or *cheating* payoff of player i is given by

$$c_i(a) = \max_{\{\tilde{a} \in A \mid \tilde{a}_j = a_j\}} g_i(\tilde{a}).$$

In the beginning of each period, each player may decide to make a monetary transfer to the other player. The players' endowment with money is assumed to be sufficiently large such that wealth constraints do not play a role. When player i and j make gross transfers $\tilde{p}_i$ and $\tilde{p}_j$, we denote by $p_i = \tilde{p}_i - \tilde{p}_j$ player i's net payment. We will generally describe payments as vectors of net payments $p = (p_1, p_2)$ with $p_1 = -p_2$. Only the player with a positive net payment makes a monetary transfer. Clearly, simultaneous transfers by both players will never be necessary to achieve a certain equilibrium payoff. Players are risk-neutral and utility is quasi-linear in money, so that player i's payoff in a period with net payments $p = (p_1, p_2)$ and action profile a is equal to $g_i(a) - p_i$.

Player i's average discounted continuation payoff at time τ given a path

$$(p^\tau, a^\tau, p^{\tau+1}, a^{\tau+1}, ...)$$

that starts in the payment stage is

$$(1 - \delta) \sum_{t=\tau}^{\infty} \delta^{t-\tau} (g_i(a^t) - p_i^t),$$

and given a path $(a^\tau, p^{\tau+1}, a^{\tau+1}, ...)$ that starts in the play stage it is

$$(1 - \delta) \sum_{t=\tau}^{\infty} \delta^{t-\tau} (g_i(a^t) - \delta p_i^{t+1}).$$

A history that ends before stage $k \in \{pay, play\}$ in period t is a list of all transfers and actions that have occurred before this point in time. Let H^k be the set of all histories that end before stage k. A strategy σ_i of player i in the repeated game maps every history $h \in H^{play}$ into an action $a_i \in A_i$, and every history $h \in H^{pay}$ into a payment. We write $\sigma|h$ for the profile of continuation strategies following history h. We denote by $u_i(\sigma|h)$ player i's average discounted payoff induced by $\sigma|h$, while $u(\sigma|h) = (u_1(\sigma|h), u_2(\sigma|h))$ denotes the vector of continuation payoffs, and $U(\sigma|h) = u_1(\sigma|h) + u_2(\sigma|h)$ the joint continuation payoff.

The set of all continuation payoffs of a given strategy profile σ at stage k is denoted by

$$\mathcal{U}^k(\sigma) = \{u(\sigma|h) : \ h \in H^k\}.$$

By Σ^k_{SGP} we mean the set of subgame perfect (continuation) equilibria that start in stage k. If σ is a subgame perfect equilibrium, we call $u(\sigma)$ a subgame perfect payoff. The set of subgame perfect payoffs at stage k is denoted by

$$\mathcal{U}^k_{SGP} = \{u(\sigma) : \ \sigma \in \Sigma^k_{SGP}\}$$

Note that we have restricted the analysis to pure strategies. Similar to Farrell and Maskin (1989) and Baliga and Evans (2000), one can allow for mixing in the stage game by letting the action space A contain all mixed strategies of the original stage game and the payoff function $g(a)$ describe the expected payoffs. It is then assumed that a player can ex post observe the other player's mixing probabilities and not only the realized outcome.

For convenience, we assume that the stage game has a Nash equilibrium in A. Our main results would also hold without this assumption as long as the discount factor δ is sufficiently large, such that a subgame perfect equilibrium of the repeated game exists.

2.2 Stationary contracts

In the following, we define a class of simple stationary strategy profiles which are helpful to characterize the Pareto frontier of subgame perfect payoffs and to study the effects of different renegotiation-proofness requirements.

Definition 1. A stationary strategy profile is characterized by a triple of action profiles (a^e, a^1, a^2), called an action plan, and a payment scheme $(p^0, p^e, F^1, F^2, f^1, f^2)$ in the following way:

In the payment stage of period 0, there are up-front payments p^0.

Whenever a player makes the prescribed payment in the payment stage, the equilibrium actions a^e are played in the next play stage.

Whenever there is no (or a bilateral) deviation from a^e, equilibrium payments p^e are conducted in the next payment stage.

If player i unilaterally deviates from a prescribed action, he pays a fine $F_i^i \geq 0$ to the other player in the subsequent payment stage.[5]

If player i deviates from a required payment, the punishment profile a^i is played in the next play stage and payments f^i are made in the subsequent payment stage.

The structure of a stationary strategy profile is illustrated in Figure 1.

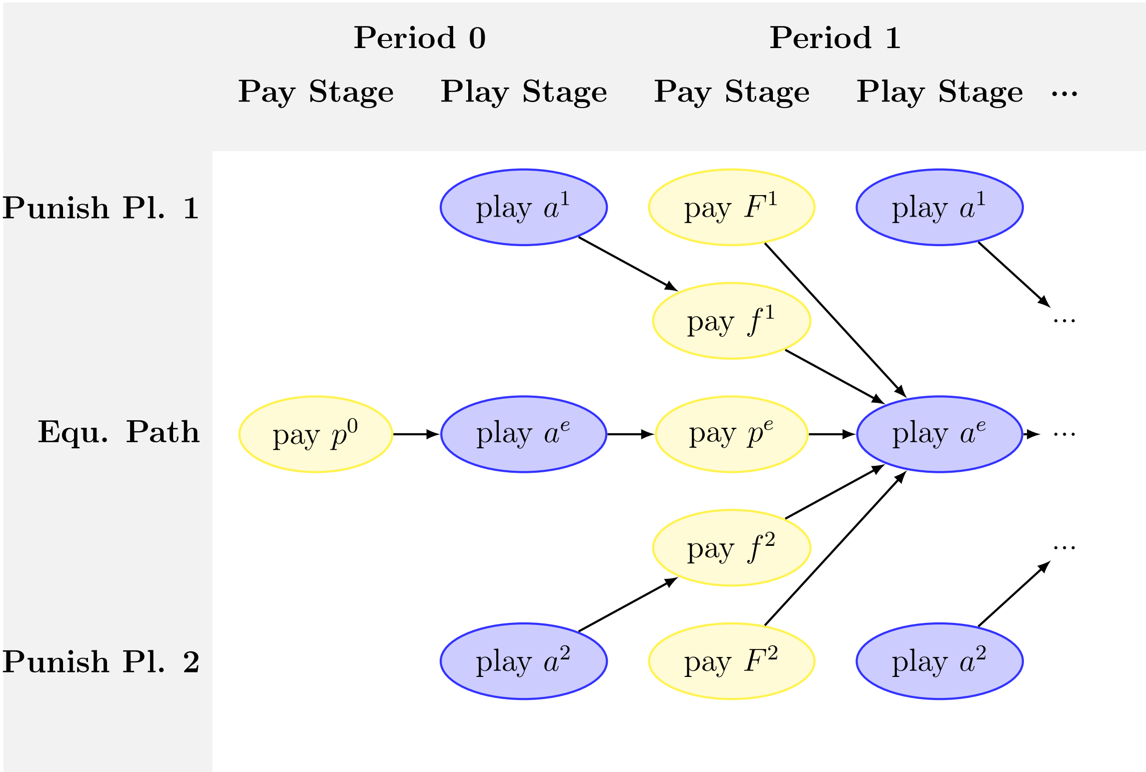

Figure 1: Structure of stationary strategy profiles. Arrows indicate continuation play if no player deviates (or a bilateral deviation takes place). If player 1 (2) unilaterally deviates then the top (bottom) row will be played in the next stage.

One can express stationary strategy profiles also in terms of simple strategies as defined by Abreu (1988). A simple strategy profile for two players prescribes play of an initial path, while any unilateral deviation from the prescribed paths by player i is followed by play along player i's punishment path. In our setting, a stationary strategy profile consists of the initial path $(p^0, a^e, p^e, a^e, p^e, ...)$ and two punishment

[5] To be consistent, we denote by F_i^i a single payment by player i and with F^i also a vector of net payments of both players with $F_j^i = -F_i^i$.

paths for each player i, depending on whether the deviation occurred in the play stage or in the payment stage: $(F^i, a^e, p^e, a^e, p^e, ...)$ resp. $(a^i, f^i, a^e, p^e, a^e, p^e, ...)$.

Abreu (1988) is built around the now familiar idea that for subgame perfection the punishment does not need to fit the crime. Any unilateral deviation from a prescribed path can be punished by the same continuation equilibrium, namely the worst possible subgame perfect equilibrium for that player. The optimal penal codes, as such worst play paths are called in Abreu's work, then often have a "stick and carrot" structure: they begin with the worst possible action for the punished player, and may reward him for complying with the punishment further along the path. In our framework, the punishment paths have a similar structure: chosen optimally, the action a^i must have a low enough cheating payoff $c_i(a^i)$ to deter a deviation by player i. The payment f_i^i is used as a lower fine that adjusts the punishment so that neither the punished player i nor the punishing player j has an incentive to deviate from the punishment profile a^i.

Recall that there are two different punishment paths because a punishment can start in the payment or play stage. Nevertheless, the intuition of Abreu (1988) goes through in the sense that both punishment paths can have the same payoff for the punished player. We fix for all stationary strategy profiles the lower fine f_i^i such that both punishment paths yield the same payoff u_i^i:

$$f_i^i = F_i^i - \frac{u_i^i - g_i(a^i)}{\delta}. \tag{1}$$

It turns out that fixing the lower fine in this way does not restrict the ability of stationary strategy profiles to characterize optimal subgame perfect and renegotiation-proof payoffs. Similarly, we will assume that an action plan always fulfills the following conditions: First, $G(a^e) \geq G(a^i)$ for both players $i = 1, 2$, i.e., the surplus created on the equilibrium path is weakly greater than the one created on the punishment paths. Second, in accordance with the "stick and carrot" intuition, the cheating payoff is lowest for the punishment action profile: we assume that for both players $i = 1, 2$ either $c_i(a^i) < c_i(a^e)$ or $a^i = a^e$.

Definition 2. A stationary strategy profile that constitutes a subgame perfect equilibrium is called a *stationary contract.*

In the following, we find conditions that imply subgame perfection of a stationary strategy profile. It is often more convenient to think about a stationary contract in terms of the continuation payoffs it defines and not in terms of the actual payments that have to be made. We denote player i's continuation payoff before a play stage on the equilibrium path by

$$u_i^e = g_i(a^e) - \delta p_i^e. \tag{2}$$

Note that because up-front payments play a special role, the equilibrium payment that belongs to an action profile is actually conducted in the next period and therefore discounted in this formula.[6] Player i's continuation payoff after deviating is called his punishment payoff and is given by

$$u_i^i = -(1-\delta)F_i^i + u_i^e. \tag{3}$$

To verify that a given stationary strategy profile is a subgame perfect equilibrium, it is sufficient to check that there are no profitable one-shot deviations. We first consider the punishment of player $i \in \{1,2\}$. Irrespective of the stage at which the punishment starts, player i's payoff is u_i^i if he complies with the punishment. If he deviates once and complies afterwards, his payoffs are u_i^i resp. $c_i(a^i)(1-\delta) + \delta u_i^i$, depending on whether the punishment started in the payment stage or the play stage. Therefore, player i will not deviate from his punishment whenever

$$u_i^i \geq c_i(a^i). \tag{4}$$

Since this implies that $u_i^i \geq g_i(a^i)$, it holds that the fine f_i^i is indeed lower than the fine F_i^i. In particular, $f_i^i = F_i^i$ holds only if player i cannot profitably deviate from a^i. Otherwise the deviator's cooperation in the punishment must be induced by the carrot of lowering the fine that is due in the next period, in extreme cases so much that the fine is actually a reward ($f_i^i < 0$).

We now turn to the role of player j in player i's punishment. We do not only have to ensure that player j does not deviate from the punishment profile a^i, but also that he pays the reward in case $f_j^i > 0$. It can be checked easily that both conditions are fulfilled if and only if

$$(1-\delta)G(a^i) + \delta G(a^e) - u_i^i \geq (1-\delta)c_j(a^i) + \delta u_j^j. \tag{5}$$

Note that the left-hand side of condition (5) is player j's continuation payoff when punishing player i at play stage. The right-hand side is his continuation payoff if he cheats in the play stage. Not cheating in the play stage (and then paying f_j^i) is always more difficult to induce than just paying f_j^i.

On the equilibrium path, compliance with both the actions a^e and the payments p^e is achieved if and only if for each player $i = 1,2$

$$u_i^e \geq (1-\delta)c_i(a^e) + \delta u_i^i. \tag{6}$$

[6] One could get rid of the discount factor in this expression by allowing two payment stages in one period, before and after an action is taken. Since a payment of p in one period is equivalent to a payment of p/δ in the next, the two formulations of the game are equivalent.

Finally, an up-front payment p^0 is subgame perfect whenever for both players $i = 1, 2$ it holds that $-p_i^0(1-\delta) + u_i^e \geq u_i^i$, which is the same as

$$p_i^0 \leq F_i^i. \tag{7}$$

By changing the up-front payment for a given equilibrium action profile and punishment payoffs, all payoffs on the line from $(u_1^1, G(a^e) - u_1^1)$ to $(G(a^e) - u_2^2, u_2^2)$ can be achieved. To summarize, a stationary strategy profile with action plan (a^e, a^1, a^2), payments p^0, p^e, and fines F^1, F^2 constitutes a stationary contract if conditions (4), (5), (6), and (7) are satisfied for both players.

Since the up-front payments can be chosen to achieve all possible initial distributions of the surplus, the set of feasible distributions is independent of the equilibrium payments p^e. The intuition is simple: if a player makes lower equilibrium payments, he is willing to make higher up-front payments that offset the distributive effects of the equilibrium payments. Hence, the payments p^e can be chosen for the sole purpose of smoothing the incentives not to deviate from the equilibrium path. In fact, whenever the sum of the two inequalities in (6) holds, then p^e can be chosen such that the individual conditions hold for both players. Furthermore, if we are merely interested in subgame perfection, we can set fines to the maximum level such that punishment payoffs are given by $u_i^i = c_i(a^i)$.[7] Making use of such an appropriate selection of equilibrium payments and fines, we can derive simple conditions for checking whether a stationary contract with some specific action plan exists.

Lemma 1. *There exists a stationary contract with action plan (a^e, a^1, a^2) if and only if*

$$G(a^e) \geq (1-\delta)(c_1(a^e) + c_2(a^e)) + \delta(c_1(a^1) + c_2(a^2)) \tag{SGP-a^e}$$

and for both players $i = 1, 2$

$$(1-\delta)G(a^i) + \delta G(a^e) - c_i(a^i) \geq (1-\delta)c_j(a^i) + \delta c_j(a^j). \tag{SGP-a^i}$$

The first condition ensures that players do not have any incentive to deviate from the equilibrium action profile a^e. The joint payoff on the equilibrium path $G(a^e)$ must be larger than the combined payoffs from cheating today and being punished in the future. Condition (SGP-a^i) ensures that no player has an incentive to deviate from the punishment of player i. The left-hand side of this condition is the continuation payoff of the punishing player j before playing a^i if maximum fines are imposed on player i. This value will also play an important role in the analysis of renegotiation-proofness. The right-hand side is player j's maximum continuation payoff if he decides to deviate instead.

[7]These maximal fines are given by $F_i^i = \frac{1}{(1-\delta)}(u_i^e - c_i(a^i))$. The maximal fines become very large as the game's surplus rises. Such extreme values are not necessary, but convenient in our search for all sustainable equilibrium payoffs.

3 Optimal Subgame Perfect Payoffs

This section shows that in our setting every Pareto-optimal subgame perfect payoff can be achieved by stationary contracts. Furthermore, we illustrate how stationary contracts allow a simple characterization of these payoffs. We denote the weak Pareto frontier of the set of subgame perfect payoffs by $\mathcal{P}^{pay}_{SGP}$.[8] Furthermore, let

$$\bar{U}_{SGP} := \sup_{u \in \mathcal{U}^{pay}_{SGP}} u_1 + u_2$$

be the supremum of joint payoffs of subgame perfect equilibria, and

$$\bar{u}^i_{SGP} := \inf_{u \in \mathcal{U}^{pay}_{SGP}} u_i$$

be the infimum of player i's payoffs. Note that these values would be the same if the range of payoffs $\mathcal{U}^{pay}_{SGP}$ was replaced by $\mathcal{U}^{play}_{SGP}$, the set of subgame perfect continuation payoffs at the play stage.

For a given discount factor, we call an action profile $\overline{a}^e$ optimal if $G(\overline{a}^e) = \bar{U}_{SGP}$ and there exists a stationary contract in which $\overline{a}^e$ is played on the equilibrium path. Similarly, we call an action profile $\overline{a}^i$ an optimal punishment for player i if $c_i(\overline{a}^i) = \bar{u}^i_{SGP}$ and there exists a stationary contract in which $\overline{a}^i$ is the punishment profile for player i. Note that a stationary contract with optimal punishments and maximal fines uses optimal penal codes in the sense of Abreu (1988). For the remainder of this paper, the labels $\overline{a}^e$ and $\overline{a}^i$ will refer to an optimal action profile and optimal punishment, respectively.

Proposition 1. *An optimal action profile $\overline{a}^e$ and optimal punishments $\bar{a}^1, \bar{a}^2$ exist. The Pareto frontier of subgame perfect payoffs is linear and can be implemented by stationary contracts with action plan $(\overline{a}^e, \overline{a}^1, \overline{a}^2)$ and maximal fines.*

Proposition 1 tells us that $\mathcal{P}^{pay}_{SGP}$ is equal to the line between $(c_1(\overline{a}^1), G(\overline{a}^e) - c_1(\overline{a}^1))$ and $(G(\overline{a}^e) - c_2(\overline{a}^2), c_2(\overline{a}^2))$. Characterizing the Pareto frontier for a given discount factor therefore boils down to finding an optimal action profile $\overline{a}^e$ and strongest punishments $\overline{a}^1, \overline{a}^2$. Using these action profiles, any Pareto-optimal subgame perfect payoff can be achieved by appropriate up-front and equilibrium payments.

To find optimal action plans, we use the conditions in Lemma 1. These conditions have a convenient structure: First, it depends only on the joint equilibrium payoff $G(a^e)$ whether an action profile a^e that satisfies (SGP-a^e) also satisfies the other two

[8]The weak Pareto frontier is defined as $\mathcal{P}^{pay}_{SGP} = \{(v_1, v_2) \in \mathcal{U}^{pay}_{SGP} : \text{if } u_1 > v_1 \text{ and } u_2 > v_2 \text{ then } u \notin \mathcal{U}^{pay}_{SGP}\}$.

conditions for given punishment action profiles a^1 and a^2. More efficient equilibrium play, i.e. a higher level of $G(a^e)$, relaxes the conditions (SGP-a^1) and (SGP-a^2), and therefore allows a larger set of action profiles to be used as punishments. Second, it depends only on the cheating payoff $c_i(a^i)$ whether an action profile a^i that satisfies condition (SGP-a^i) also satisfies the other two conditions for given a^j and a^e. A stronger punishment of player i, i.e. a lower level of $c_i(a^i)$, relaxes the conditions (SGP-a^e) and (SGP-a^j), and thereby facilitates a stronger punishment of player j and more efficient equilibrium play. In this sense, higher joint payoffs and harsher punishments complement each other.

Consequently, there is a simple iterative procedure that yields a list of optimal action profiles and optimal punishments for all discount factors for stage games with finite action spaces. This procedure works as follows: For each round $n = 0, 1, 2, ...$ we define an action plan $(a^e(n),\, a^1(n),\, a^2(n))$, starting with

$$a^e(0) \in \arg\max_{a \in A} G(a)$$

and

$$a^i(0) \in \arg\min_{a \in A} c_i(a).$$

This means we start with the most efficient action profile and the harshest punishments of the stage game. Assume that if there are multiple candidates for $a^e(0)$, we pick the candidate with the lowest sum of cheating payoffs. Then the existence of a stage game Nash equilibrium guarantees that there exists a critical discount factor $\delta(0) < 1$ such that for every discount factor $\delta \in [\delta(0), 1)$, the profiles $a^e(0)$,$a^1(0)$,$a^2(0)$ satisfy (SGP-a^e) and both (SGP-a^i) and constitute an optimal action plan.

The following definitions are used to recursively define the action plan for subsequent rounds. For each $k \in \{e, 1, 2\}$ and each round $n \geq 0$, let the variable $\delta^k(n)$ denote the minimum discount factor for which the action plan of round n fulfills condition (SGP-a$^{\mathrm{k}}$), and denote the binding critical discount factor by $\delta(n) = \max_k \delta^k(n)$.[9] We denote by $\delta^*(n) = \min_{m \leq n} \delta(m)$ the smallest critical discount factor that has been found in round n or earlier rounds.

We say an action profile a^k relaxes the condition (SGP-a$^{\mathrm{k}}$) of round n if replacing $a^k(n)$ by a^k (keeping the other action profiles unchanged) makes condition (SGP-a$^{\mathrm{k}}$) hold for a larger set of discount factors than $[\delta^*(n), 1)$. Let $R^k(n)$ denote the set of action profiles that relax condition (SGP-a$^{\mathrm{k}}$) of round n. If $\delta^e(n) \geq \delta^*(n)$, we choose in round n an action profile

$$a^e(n+1) \in \arg\max_{a \in R^e(n)} G(a),$$

[9] We normalize $\delta^k(n) = 1$ if no discount factor $\delta < 1$ fulfills condition (SGP-a$^{\mathrm{k}}$).

which relaxes condition (SGP-a^{e}) and has a joint payoff as large as possible. Similarly, if $\delta^i(n) \geq \delta^*(n)$, we choose an action profile

$$a^i(n+1) \in \arg\min_{a \in R^i(n)} c_i(a),$$

which relaxes condition (SGP-a^{i}) and has a cheating payoff as small as possible. If $\delta^k(n) < \delta^*(n)$, we keep the old action profile for round k, i.e. $a^k(n+1) = a^k(n)$. The procedure stops once the binding constraint cannot be relaxed anymore.

Proposition 2. *Consider the procedure above for a finite action space A. If in round n the critical discount factor falls below the previous minimum, i.e. $\delta(n) < \delta^*(n-1)$, then $(a^e(n), a^1(n), a^2(n))$ is an optimal action plan for all discount factors $\delta \in [\delta(n), \delta^*(n-1))$. The procedure terminates in a finite number of rounds and yields optimal action plans for all discount factors $\delta \in [0,1)$.*

It is shown in the proof of Proposition 2 that $G(a^e(n))$ weakly decreases and $c_i(a^i(n))$ weakly increases with the number of rounds n. This has the consequence that when one condition (SGP-a^{k}) is relaxed by replacing the action profile $a^k(n-1)$, the other conditions are tightened. It can therefore well be the case that the critical discount factor $\delta(n)$ increases between some rounds. If this is the case, one simply proceeds to the next round; no new optimal action plan was found in round n.

Even though the critical discount factor $\delta(n)$ does not monotonically decrease, the algorithm always stops in a finite number of rounds. The reason is that -as is shown in the proof of Proposition 2- in every round and for all $k \in \{e, 1, 2\}$ the set of action profiles $R^k(n)$ that relax condition (SGP-a^{k}) becomes weakly smaller and strictly smaller for at least one $k \in \{e, 1, 2\}$.

In the final round all optimal action profiles will be stage game Nash equilibria. Players then do not have incentives to deviate even if no weight is attached to payoffs in future periods; the critical discount factor will be zero.

3.1 Example: Simplified Cournot Game due to Abreu

We now illustrate the procedure above, for a simplified Cournot game taken from Abreu (1988). Two firms simultaneously choose either low (L), medium (M), or high (H) output, and stage game payoffs are given by the following matrix:

		Firm 2		
		L	M	H
	L	10,10	3,15	0,7
Firm 1	M	15,3	7,7	-4,5
	H	7,0	5,-4	-15,-15

Joint payoffs are maximized if firms choose (L,L), the unique Nash equilibrium of the stage game is (M, M), and high output minimizes the cheating payoff of the other firm. Abreu considered the case without side-payments and constructed optimal penal codes that support collusive play of (L, L) for any discount factor $\delta \geq \frac{4}{7}$, while the threat of an infinite repetition of the stage game equilibrium can sustain collusion only if $\delta \geq \frac{5}{8}$.

For the case with side payments, the first candidate for an optimal action profile is clearly the collusive outcome, i.e. $a^e(0) = (L, L)$. A harshest punishment of the stage game requires that the punisher chooses high output. Of all such action profiles, we only need to consider the action profile that maximizes $G(a^i) - c_j(a^i)$ in order to relax condition (SGP-a$^{\text{i}}$) as much as possible. We therefore choose $a^1(0) = (M, H)$ and $a^2(0) = (H, M)$. While it would be more efficient for both players if the punished player chooses low output, the choice of medium output substantially reduces the punishers' incentives to deviate from the punishment and therefore makes the punishment easier to implement.

With this action plan, condition (SGP-a^e) holds for all $\delta \geq \frac{1}{3}$ while conditions (SGP-a^i) hold for all $\delta \geq \frac{3}{13}$. Hence, the collusive outcome can be sustained for all discount factors $\delta \geq \frac{1}{3}$. To characterize Pareto-optimal payoffs for lower discount factors, we relax condition (SGP-a^e) by choosing either $a^e(1) = (L, M)$ or $a^e(1) = (M, L)$ and keep the previous punishment profiles. Condition (SGP-a^e) then holds for all $\delta \geq \frac{2}{11}$ while conditions (SGP-a^i) hold for all $\delta \geq \frac{1}{4}$. Thus, for all $\delta \in [\frac{1}{4}, \frac{1}{3})$, a partial collusive equilibrium play of (L, M) or (M, L) can be sustained. Note that the corresponding stationary contracts require positive equilibrium payments from the firm that chooses medium output to the firm that chooses low output.[10] Continuing the procedure, we find that for lower discount factors only an infinite repetition of the stage game equilibrium can be sustained.

One can see from this example how side payments are useful in a number of ways. First, they are a way to get around asymmetries in the stage game. While in this symmetric game equilibrium payments are not needed to implement action profiles on the diagonal, they are needed to implement (L, M).

Second, side payments can be used to fine-tune the punishment payoff to the discount factor. To see what this means, consider for example punishment paths without payments that repeat play of a punishment action like (H, M) for a number of periods, and then return to play of (L, L) forever. The problem with these punishment paths is that the number of repetitions is an integer, while the discount factors vary smoothly. This problem can be met by introducing a correlation device, and indeed collusion would also be implementable for $\delta \geq \frac{1}{3}$ if there was a correlation

[10] Using condition (6) (see Section 2.2) one finds that these payments have to lie in the interval $[\frac{4-7\delta}{\delta}, 15]$.

device instead of side payments in this game. For example, for $\delta = \frac{1}{3}$ the punishment would consist of playing (H, M) once and returning to play of (L, L) for the rest of the game with probability 0.8, while with probability 0.2 the punishment is started again.

The third way in which the ability to transfer utility matters is by achieving all distributions of payoffs with an up-front payment, which is not possible without side payments.

3.2 Example: Prisoners' Dilemma

For another example, consider a prisoner's dilemma game with payoff matrix:

		Player 2 C	Player 2 D
Player 1	C	$(1, 1)$	$(S-d, d)$
	D	$(d, S-d)$	$(0, 0)$

where $d > 1 > \frac{S}{2}$. The first candidates for optimal punishments are $\overline{a}^1 = \overline{a}^2 = (D, D)$. Since this punishment is a Nash equilibrium of the stage game, conditions (SGP-a^{i}) hold for all discount factors. The first candidate for optimal equilibrium actions is mutual cooperation (C, C) and condition (SGP-a^{e}) shows that it can be sustained for all $\delta \geq \frac{d-1}{d}$. If $S > 1$ we find that the asymmetric equilibrium action profiles (C, D) and (D, C) are optimal for all $\delta \in [\frac{d-S}{d}, \frac{d-1}{d})$. For lower discount factors, only the Nash equilibrium of the stage game (D, D) can be sustained.

We compare this result to the setting without side payments. If, as in this game, the lowest cheating payoff can be attained by an equilibrium of the stage game, then optimal penal codes consist of an infinite repetition of this equilibrium, independent of whether side-payments are possible or not. The analysis in Abreu (1988) hence implies that the set of subgame perfect payoffs can be found by considering all paths of play that players will adhere to given a punishment of (D, D) forever. If one wants to implement mutual cooperation (C, C) in every period, side payments on the equilibrium path provide no help. Thus the critical discount factor under the well known grim-trigger strategies without side payments is the same as under optimal strategies with side payments. However, to find the Pareto frontier of subgame perfect payoffs without side payments, a large number of other paths would have to be considered. How complicated this can be becomes clear in Sorin (1986), who calculates the set of attainable payoffs for one example of a prisoner's dilemma game with mixed strategies and a fixed discount factor. In this example, the Pareto frontier consists of only three points. The difficulties are also well illustrated by Mailath et al. (2002).

4 Strong Optimality and Strong Perfection

4.1 Definitions and Main Results

In many relationships it seems reasonable that players have the possibility to meet and renegotiate their existing relational contract. If players anticipate such a renegotiation, a subgame-perfect equilibrium may cease to be stable, however. There are different concepts of renegotiation-proofness that intend to refine the set of subgame perfect equilibria to those equilibria that are robust against this criticism.

We first consider the concept of *strong optimality* that Levin (2003) applies in his study of repeated principal-agent games. Levin implicitly assumes that renegotiation can only take place at the beginning of a period, i.e. before the payment stage but not before the play stage. A minimum requirement for a successful renegotiation at this stage is that there exists a new contract that is subgame perfect and creates some surplus compared to the existing contract, in the sense of achieving a higher joint payoff. Consequently, there is never scope for renegotiation if all continuation equilibria already achieve the highest joint payoff $\overline{U}_{SGP}$ that is possible in a subgame perfect equilibrium.

Definition 3. A subgame perfect equilibrium σ is *strongly optimal* (with respect to renegotiations at payment stages) if $U(\sigma|h) = \overline{U}_{SGP}$ for all $h \in H^{pay}$.

Proposition 1 directly implies the following result:

Corollary 1. *Every stationary contract with optimal equilibrium action profile $\overline{a}^e$ is strongly optimal. The Pareto frontier of subgame perfect and strongly optimal payoffs coincide.*

The reason why every stationary contract with optimal equilibrium actions is strongly optimal is that in every continuation equilibrium starting at a payment stage, the required payments will always be conducted and afterwards the optimal action profile $\overline{a}^e$ is played in all subsequent periods. Since by assumption there is no renegotiation directly before a play stage, continuation equilibria that require play of a punishment profile a^i are never subject to renegotiation.

A more stringent test of renegotiation considers the possibility of renegotiation at all stages within a period. This is assumed, for example, by Fong and Surti (2009) who study repeated prisoner's dilemma games with side payments. The strictest concept of renegotiation-proofness would be a modification of strong optimality that requires for every continuation equilibrium — including those starting at a payment stage— that the sum of continuation payoffs is equal to the highest possible value $\overline{U}_{SGP}$.

Since punishment actions typically require some efficiency loss, this condition is too strong to allow for much insightful analysis. A slightly weaker requirement follows from adapting *strong perfection* (see Rubinstein, 1980) to our set-up.

Definition 4. A subgame perfect equilibrium σ is *strong perfect* at both stages if $\mathcal{U}^k(\sigma) \subset \mathcal{P}^k_{SGP}$ for all $k \in \{pay, play\}$.

Strong perfection requires for both stages that no continuation payoff is strictly Pareto-dominated by another subgame perfect continuation payoff of the same stage. Strong perfect equilibria may fail to exist, but the concept provides a useful sufficient condition for renegotiation-proofness. If there is no subgame perfect equilibrium that makes both players better off, then one may feel confident that renegotiation is deterred. Let $\overline{u}^i_{SP}$ denote the infimum of player i's payoffs in strong perfect equilibria, in case such equilibria exist.

Proposition 3. *Every strong perfect payoff can be implemented by a strong perfect stationary contract with optimal equilibrium actions $\bar{a}^e$. The set of strong perfect payoffs is either empty or given by the line from $(\overline{u}^1_{SP}, G(\bar{a}^e) - \overline{u}^1_{SP})$ to $(G(\bar{a}^e) - \overline{u}^2_{SP}, \overline{u}^2_{SP})$.*

A strong perfect stationary contract clearly requires optimal equilibrium actions $\bar{a}^e$, but it is not generally the case that optimal penal codes can be used. We now derive results that help to find strong perfect stationary contracts for a given game and discount factor or to verify their non-existence.

In a stationary contract with optimal equilibrium action profile, only the continuation equilibrium starting at the play stage in the punishment phase can be dominated by another subgame perfect continuation equilibrium. To find the set of strong perfect stationary contracts, one therefore needs to understand the structure of the Pareto frontier of subgame perfect payoffs in the play stage. Similar to the result that the Pareto frontier of payoffs in the payment stage can be found by varying incentive compatible up-front payments in an optimal stationary contract, one can show that the Pareto frontier of payoffs in the play stage can be found by varying the first action profile (and subsequent payment) in an optimal stationary contract.

To formalize this idea, we define an *auxiliary contract* as an equilibrium that coincides with an optimal stationary contract except that on the equilibrium path there are no upfront payments, the action profile in the first play stage can differ from $\bar{a}^e$, and the payments in the second period are adjusted such that subgame perfection is still satisfied. For example, all continuation equilibria of an optimal stationary contract that start in the play stage are auxiliary contracts. We say action profile $\tilde{a}$ is *admissible* if there exists an auxiliary contract with first period action profile $\tilde{a}$.

We get the following characterization of admissible action profiles and auxiliary contracts.

Lemma 2. *An action profile $\tilde{a}$ is admissible if and only if*

$$(1-\delta)G(\widetilde{a}) + \delta G(\overline{a}^e) \geq (1-\delta)(c_1(\widetilde{a}) + c_2(\widetilde{a})) + \delta(c_1(\overline{a}^1) + c_2(\overline{a}^2)). \tag{8}$$

The set of payoffs that can be achieved with auxiliary contracts with first action profile $\tilde{a}$ is the line segment

$$\{u | u_1 + u_2 = (1-\delta)G(\tilde{a}) + \delta G(\bar{a}^e) \text{ and } u_i \geq (1-\delta)c_i(\tilde{a}) + \delta c_i(\bar{a}^i)\} \tag{9}$$

Auxiliary contracts allow us to describe the Pareto frontier of subgame perfect continuation equilibria at play stage.

Proposition 4. *The Pareto frontier of subgame perfect continuation equilibria at play stage is the Pareto frontier of all auxiliary contract payoffs.*

For an example, consider the repeated prisoner's dilemma game from Section 3.2 with parameter values $d = 2.5$, $S = 0$ and $\delta = \frac{2}{3}$. The optimal action plan is $\bar{a}^e = (C, C)$ and $\bar{a}^1 = \bar{a}^2 = (D, D)$, and only (C, C) and (D, D) are admissible. Figure 2 shows the payoffs that can be implemented with auxiliary contracts for each of the admissible action profiles.

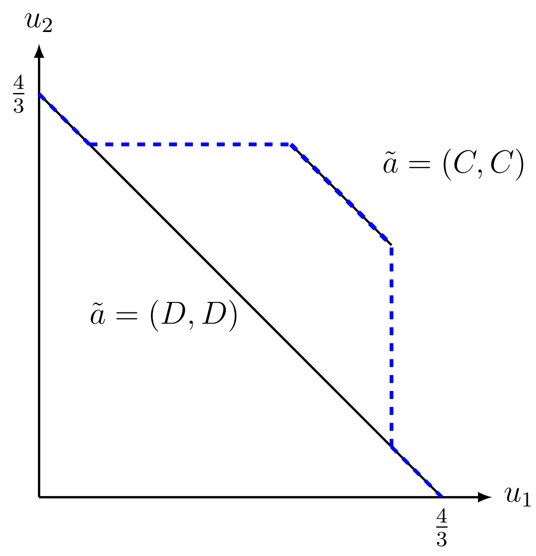

Figure 2: Example of Pareto frontier at play stage for a prisoner's dilemma.

The Pareto frontier of subgame perfect continuation payoffs before play stage is indicated by the dashed curve. It consists of the payoffs of auxiliary contracts with $\tilde{a} = (C, C)$ and parts on the left and right end of the payoffs of the auxiliary contracts with $\tilde{a} = (D, D)$. The payoff line for auxiliary contracts starting with (D, D) has a lower level but a wider span than the line corresponding to (C, C). The intuition for the wider span is that transfers after (C, C) must be such that no player has an incentive to deviate from (C, C) while no such restriction needs to be satisfied if one starts with (D, D). Here the difference in the spans for (C, C) and (D, D) outweighs the difference in levels so that the end points of the (D, D) line are Pareto optimal.

This means that in this example an optimal stationary contract with maximum fines is strongly perfect, since a punishing player has a higher expected continuation payoff at play stage before the punishment than in any continuation equilibrium that starts with play of (C, C). Intuitively, one may think that it should be possible to restrict attention to stationary contracts with maximum fines in order to implement any strong perfect payoff since in order to block renegotiation it mainly seems important to reduce the punisher's incentives to renegotiate a punishment. Indeed, in many examples this intuition will be true. Yet, we illustrate in Appendix A that there exist cases in which strong perfection can only be achieved if fines are below their maximum level in order to increase the incentives of the punished player to block renegotiation before play of a^i. In general, we have the following result:

Proposition 5. *A stationary contract with action plan* $(\bar{a}^e, a^1, a^2)$ *and punishment payoffs* u_1^1 *and* u_2^2 *is strong perfect if and only if for both players* $i = 1, 2$ *and for all admissible* $\tilde{a}$ *with* $G(\tilde{a}) > G(a^i)$ *it holds that either*

$$(1-\delta)G(a^i) - u_i^i \geq (1-\delta)\left(G(\tilde{a}) - c_i(\tilde{a})\right) - \delta c_i(\bar{a}^i) \text{ or} \tag{SP1}$$

$$u_i^i \geq (1-\delta)(G(\tilde{a}) - c_j(\tilde{a})) + \delta G(\bar{a}^e) - \delta c_j(\bar{a}^j) \tag{SP2}$$

Conditions (SP1) and (SP2) concern the punishment for player i at play stage. Condition (SP1) ensures that there exist no subgame perfect continuation equilibria that give a higher payoff to the punishing player j, i.e., that the punisher has no incentive to renegotiate the punishment. Should such continuation equilibria exist, condition (SP2) ensures that they would make the punished player i worse off.

While Proposition 5 states necessary and sufficient conditions for strong perfection of a stationary contract, these conditions contain the punishment payoffs and can therefore be tedious to apply. We therefore provide some corollaries that make the application of Proposition 5 easier to apply. For the examples below, the following corollary provides sufficient conditions.

Corollary 2. *A stationary contract with an optimal action plan* $(\bar{a}^e, \bar{a}^1, \bar{a}^2)$ *and maximal fines is strong perfect if for both players* $i = 1, 2$ *and all admissible action profiles* $\tilde{a}$ *with* $G(\tilde{a}) > G(\bar{a}^i)$ *it holds that*

$$G(\bar{a}^i) - c_i(\bar{a}^i) \geq G(\tilde{a}) - c_i(\tilde{a}). \tag{10}$$

As becomes clear in this condition, there are two things that help to make a stationary contract strong perfect, both allowing the punishing player to get a large payoff: one is that the punishment profile is jointly very efficient, and the other is that the punishment is very effective by giving a low best-reply payoff to the punished player. This corollary is very helpful because once one has found an optimal stationary contract with this property, it must be the case that the set of strong perfect payoffs and the Pareto frontier of subgame perfect payoffs coincide. The corollary can therefore serve as a quick way to check robustness against renegotiation.

The following corollary of Proposition 5 is helpful for showing that a given stationary contract is not strong perfect.

Corollary 3. *An action profile a cannot be used as the punishment profile in a strong perfect stationary contract if there exists an admissible action profile $\tilde{a}$ with $G(\tilde{a}) > G(a)$ as well as for both $i = 1, 2$*

$$G(\tilde{a}) - c_i(\tilde{a}) > G(a) - c_i(a). \tag{11}$$

While this corollary tells us that a particular action profile cannot be used as punishment at all, the next corollary can only be used for pairs of punishment profiles:

Corollary 4. *A stationary contract with action plan $(\bar{a}^e, a^1, a^2)$ cannot be strong perfect if for both players $i = 1, 2$ we have $G(\bar{a}^e) > G(a^i)$ and*

$$(1-\delta)\left(G(\bar{a}^e) - c_i(\bar{a}^e)\right) - \delta c_i(\bar{a}^i) > (1-\delta)G(a^i) - c_i(a^i). \tag{12}$$

4.2 Example: Strong Perfection in the Prisoners' Dilemma

Recall the repeated prisoner's dilemma game from the previous section. Using Lemma 2, we find that whenever $S < 1$ and $\frac{d-1}{d} \leq \delta < \frac{d-S}{d-S+2}$ then only (C, C) and (D, D) are admissible. We have $G(C, C) - c_i(C, C) = 2 - d$ and $G(D, D) - c_i(D, D) = 0$ and thus find from Corollary 2 that all optimal stationary contracts are strong perfect if $d \geq 2$. If $d < 2$, we see from Corollary 3 that (D, D) can not be part of a strong perfect stationary contract, using $\tilde{a} = (C, C)$. The set of strong perfect payoffs is then empty. Larger values of d make deviations from (C, C) more attractive and thereby decrease the span of the payoff line for auxiliary contracts starting with (C, C). A punisher's incentives to renegotiate a punishment are thus reduced.

Using Lemma 2, we can get the parameter ranges for which alternative combinations of action profiles are admissible. If (C, D) and (D, C) are admissible, they must be used as punishments in a strong perfect equilibrium if their joint payoff satisfies $S > 0$, while if $S < 0$ one must punish with (D, D). Higher levels of S generally facilitate strong perfection. If all four action profiles are admissible, strong perfect equilibria exist if and only if $d \geq 2 - \max(S, 0)$. For the case that (C, D) and (D, C) are admissible but not (C, C), any stationary contract with action profiles in the set $\{(C, D), (D, C)\}$ must be strong perfect as there are no admissible action profiles with a larger joint payoff. If only (D, D) is admissible, the infinite repetition of the stage game Nash equilibrium is trivially strong perfect.

4.3 Example: Principal-Agent Game

We now illustrate strong perfection in a principal agent game. Assume that only player 1 (the agent) chooses an action $a \in A = [0, a_{max}]$ with $a_{max} > 0$. The action creates a nonpositive payoff $g_1(a) \leq 0$ for the agent and a weakly positive benefit $g_2(a) \geq 0$ for player 2, the principal. One interpretation is that player 1 is a supplier who delivers a product of a certain quality, where higher quality is more expensive. Another interpretation is that player 1 is a worker who can exert work effort a, which can be observed by the employer. We also assume that the agent can choose a 'do-nothing' action $a = 0$ that yields zero payoff for both players.

Since the principal gets a nonnegative benefit, $\overline{a}^2 = 0$ is an optimal punishment of the principal. Since the agent's cheating payoff in the play stage is always 0, every action $\overline{a}^1 \in A$ is by definition an optimal punishment of the agent. In particular, also the optimal equilibrium action $\overline{a}^e$ constitutes an optimal punishment of the agent. Using these punishments, we find from conditions (SGP-a$^{\mathrm{e}}$) and (SGP-a$^{\mathrm{i}}$) that the optimal equilibrium action $\overline{a}^e$ solves $\max_{a^e \in A} G(a^e)$ subject to $\delta g_2(a^e) \geq -g_1(a^e)$.

Using Corollary 2, we find that the stationary contracts with action plan $(\overline{a}^e, \overline{a}^e, 0)$ are strong perfect if for every admissible $\tilde{a}$ with $G(\tilde{a}) > 0$ the condition

$$G(\tilde{a}) - g_2(\tilde{a}) \leq 0$$

holds. Since $g_1(\tilde{a}) \leq 0$, this condition is always fulfilled. Hence, in this simple complete information game, we confirm the intuition of Levin (2003) that when the incentive problem is one-sided, optimal subgame perfect payoffs can be implemented in a renegotiation-proof way.

4.4 Example: Abreu's Simple Cournot Game

We will show that in Abreu's Cournot example there is no strong perfect equilibrium, except for the Nash equilibrium of the stage game in case $\delta < \frac{1}{4}$.

First, we use Corollary 3 to see which action profiles may at all be used as punishments in a strong perfect stationary contract. The only way in which this corollary depends on the discount factor is via the admissibility conditions. We take

$\tilde{a} = (M, M)$ which is admissible for all δ. The conditions let us easily exclude the use of any action profile a^i with $c_i(a^i) = 0$ as a punishment.

Consider now the case $\delta \geq \frac{1}{3}$, in which any strong perfect stationary contract must have the equilibrium action profile $\bar{a}^e = (L, L)$. We use Corollary 4 to exclude the action profiles a^i with $c_i(a^i) = 7$ as punishments. Since for this range of discount factors it holds that $c_i(\bar{a}^i) = 0$, the lefthand side of condition (12) equals $(1 - \delta)5$. For $a^i = (M, M)$ the right-hand side equals $7 - 14\delta$, which makes the condition true for all $\delta > \frac{2}{9}$, which is implied by $\delta \geq \frac{1}{3}$. Finally, for $a^1 = (L, M)$ or $a^2 = (M, L)$ the right-hand side equals $11 - 18\delta$, which makes the condition true for $\delta > \frac{6}{13}$, which is greater than $\frac{1}{3}$. However, a punishment payoff of 7 for both players suffices to induce collusion only if $\delta \geq \frac{5}{8}$. Finally, consider the case that one of the punishment profiles, say a^1, is (L, L). In this case it must hold that the other punishment has $c_2(a^2) = 0$, which cannot be true in a strong perfect stationary contract.

Similarly, for the case $\delta \in [\frac{1}{4}, \frac{1}{3})$, we have $\bar{a}^e = (L, M)$ and in any stationary contract with that equilibrium action profile at least one of the punishment profiles must have $c_i(a^i) = 0$.

5 Weak and Strong Renegotiation-Proofness

5.1 Definitions and Main Results

Strong perfection is a very strict criterion; in a strong perfect equilibrium every continuation payoff must survive comparison to all subgame perfect equilibria, including those that are not renegotiation-proof themselves. In fact, strong perfect equilibria often fail to exist, because punishment payoffs are Pareto dominated by a cooperative continuation equilibrium that itself requires the dominated punishment as a threat. While strong perfection is thus a sensible sufficient condition for renegotiation-proofness, it is too strict as a neccessary condition.

In this section, we analyze two concepts that only consider renegotiation to continuation equilibria that are in themselves renegotiation-proof, namely weak and strong renegotiation-proofness defined by Farrell and Maskin (1989). An equilibrium is weakly renegotiation-proof if none of its own continuation equilibria is strictly Pareto dominated by another one of its continuation equilibria. Strong renegotiation-proofness requires stability against renegotiation to any weakly renegotiation-proof continuation equilibrium. The formal definitions, allowing for renegotiation within a period, are as follows:

Definition 5. A subgame perfect equilibrium σ is weakly renegotiation-proof (WRP), if for no stage k there are two continuation payoffs $u, u' \in \mathcal{U}^k(\sigma)$ such that u is strictly Pareto-dominated by u'.

WRP equilibria always exist but the concept often does not have much restricting power. For example, it is always a WRP equilibrium to play the same Nash equilibrium of the stage game in every period and never conduct any payments.

Let $\mathcal{U}^k_{WRP}$ denote the set of all WRP (continuation-)payoffs of stage k.

Definition 6. A WRP equilibrium σ is strongly renegotiation-proof (SRP) if for no stage k and $u \in \mathcal{U}^k(\sigma)$ there exists another weakly renegotiation-proof payoff $u' \in \mathcal{U}^k_{WRP}$ such that u is strictly Pareto-dominated by u'.

It follows directly from this definition that the set of SRP payoffs is a subset of the Pareto frontier of all WRP payoffs, but in general the two sets do not coincide. In fact, for intermediate discount factors SRP equilibria often do not even exist.

We first establish conditions for existence of stationary contracts that are WRP. We then show that if the discount factor is at least $\frac{1}{2}$, one can restrict attention to stationary contracts in order to characterize the Pareto frontier of WRP payoffs and the set of SRP payoffs. Afterward, we illustrate what can go wrong for smaller discount factors.

In a stationary contract, we only have to worry about renegotiation before the play stage in a punishment phase. Furthermore, our regularity conditions for stationary contracts ($G(a^e) \geq G(a^i)$ and either $c_i(a^i) < c_i(a^e)$ or $a^i = a^e$) ensure that only renegotiation to the equilibrium path can become a problem. The following lemma shows that weak renegotiation-proofness is equivalent to the condition that the continuation payoff of the punishing player j at the play stage of player i's punishment is as least as high as player j's payoff at the play stage on the equilibrium path.

Lemma 3. *A stationary contract with action plan (a^e, a^1, a^2), maximum fines, and equilibrium payoffs u^e is WRP if and only if for both players $i = 1, 2$*

$$(1-\delta)G(a^i) + \delta G(a^e) - c_i(a^i) \geq u^e_j \tag{13}$$

Condition (13) still contains the equilibrium payment. Ideally, we would like to have a condition only on the action profiles. A simple condition can be derived if the stage game is symmetric, and one wants to check weak renegotiation-proofness for stationary contracts that have a symmetric action plan (a^e, a^1, a^2), where symmetry means $a^e_1 = a^e_2$, $a^1_1 = a^2_2$ and $a^1_2 = a^2_1$. In this case both the subgame perfection and renegotiation constraints are best balanced between players by not making any equilibrium payments, i.e. $p^e = 0$. Lemma 3 then yields the following result.

Remark 1. *If the stage game is symmetric and there are stationary contracts with a symmetric action plan (a^e, a^1, a^2), there exist WRP stationary contracts with that action plan if and only if*

$$(1-\delta)G(a^1) - c_1(a^1) \geq (1-2\delta)\, g_1(a^e). \tag{14}$$

Proposition 6 allows a characterization also for asymmetric games or action plans.

Proposition 6. *If the set of stationary contracts with action plan (a^e, a^1, a^2) is non-empty, it contains WRP contracts with maximum fines if and only if for $i = 1, 2$*

$$(1-\delta)G(a^i) + \delta G(a^e) - c_i(a^i) \geq (1-\delta)c_j(a^e) + \delta c_j(a^j) \tag{WRP-i}$$

$$(1-\delta)(G(a^1) + G(a^2)) + (2\delta - 1)G(a^e) \geq c_1(a^1) + c_2(a^2). \tag{WRP-Joint}$$

Equilibrium payments now have to perform two kinds of balancing acts: balancing subgame perfection incentives and balancing renegotiation-proofness incentives. The left-hand side of (WRP-i) denotes player j's continuation payoff at player i's punishment phase: high values facilitate renegotiation-proofness. The right-hand side denotes player j's continuation payoffs when optimally defecting in the equilibrium phase: low values allow to freely distribute the payoff on the equilibrium path and thereby facilitate renegotiation-proofness.

Condition (WRP-joint) is simply the sum of the WRP condition (13) for both players. Note that the level of joint equilibrium payoffs $G(a^e)$ has ambiguous effects on renegotiation-proofness. If joint payoffs increase, renegotiation-proofness is harder to satisfy since the equilibrium path becomes more attractive. Yet, renegotiation-proofness also becomes easier to satisfy because larger fines will be incentive compatible, which means a higher continuation payoff can be given to a punisher. The condition shows that the latter positive effect prevails whenever the discount factor satisfies $\delta \geq \frac{1}{2}$. For the intuition behind this threshold consider the effect of an increase of $G(a^e)$ by 1 on the incentives to renegotiate punishment: using transfers, such an increase can be split equally on the equilibrium path among both players such that renegotiation incentives for each player increase only by $\frac{1}{2}$, whereas fines can be increased by δ, i.e. the punishers' incentives to renegotiate are reduced by δ.

For the case $\delta \geq \frac{1}{2}$, stationary contracts allow us to characterize the Pareto frontier of WRP payoffs:

Proposition 7. *Let $\delta \geq \frac{1}{2}$. Every payoff on the Pareto frontier of WRP payoffs can be implemented with a WRP stationary contract with maximum fines.*

Different from the case of strong perfection and subgame perfection, the Pareto-frontier of WRP payoffs is not necessarily a line. This will be true again for the set strongly renegotiation-proof payoffs. We say an action profile $\widehat{a}^e$ is *WRP-optimal* if there exists a WRP stationary contract in which $\widehat{a}^e$ is played on the equilibrium path, and which implements the highest sum of payoffs that can be implemented with a WRP equilibrium. Let $\overline{u}^i_{SRP}$ denote the infimum of player i's payoffs in SRP equilibria, if such equilibria exist.

Proposition 8. *Let $\delta \geq \frac{1}{2}$. Every SRP payoff can be implemented by a stationary contract with WRP-optimal equilibrium action profile $\widehat{a}^e$. The set of SRP payoffs is either empty or given by the line from $(\overline{u}^1_{SRP}, G(\widehat{a}^e) - \overline{u}^1_{SRP})$ to $(G(\widehat{a}^e) - \overline{u}^2_{SRP}, \overline{u}^2_{SRP})$.*

While this proposition tells us that the set of SRP payoffs can be described by an SRP stationary contract, it does not tell us how to find this stationary contract and whether it exists at all. To answer such questions, one can use the following sufficient condition for an optimal WRP stationary contract to be SRP, which checks whether at play stage a punisher is at least as well off as he would be when punishing in any other WRP stationary contract with maximal fines.

Proposition 9. *Let $\delta \geq \frac{1}{2}$. A WRP stationary contract with maximum fines, WRP-optimal equilibrium action profile $\widehat{a}^e$ and punishment profiles a^1, a^2 is SRP if for both players $i = 1, 2$ and all action plans $(\tilde{a}^e, \tilde{a}^1, \tilde{a}^2)$ that satisfy (WRP-i) and (WRP-joint)*

$$(1-\delta)G(a^i) + \delta G(\widehat{a}^e) - c_i(a^i) \geq (1-\delta)G(\tilde{a}^i) + \delta G(\tilde{a}^e) - c_i(\tilde{a}^i). \tag{15}$$

5.2 Remarks on the case $\delta < \frac{1}{2}$

Recall that if $\delta < \frac{1}{2}$, then condition (WRP-Joint) in Proposition 6 is relaxed for lower joint equilibrium payoffs $G(a^e)$. This is due to the two-sided nature of weak renegotiation-proofness: while one way to avoid renegotiation of punishments is to choose more efficient punishments, which guarantee the punisher a higher payoff than on the equilibrium path, another way is to choose sufficiently inefficient equilibrium play. Since the other subgame perfection and WRP conditions require a sufficiently large joint equilibrium payoff $G(a^e)$, it may then be optimal to have a small degree of inefficiency that requires to alternate between different action profiles. While for $\delta \geq \frac{1}{2}$ our results ensure that such alternation is not required to achieve Pareto-optimal WRP or SRP payoffs, we will show with the following example that this does not generally hold true for $\delta < \frac{1}{2}$.

Let the stage game be a prisoner's dilemma with payoff matrix

	C	D
C	$(1,1)$	$(-\frac{6}{10}, \frac{11}{10})$
D	$(\frac{11}{10}, -\frac{6}{10})$	$(0,0)$

A stationary contract with $a^e = (C,C)$ exists whenever $\delta \geq \frac{1}{11}$. In a WRP stationary contract mutual cooperation can be sustained if and only if $\delta \geq \frac{1}{3}$.[11] This critical discount factor is determined by condition (WRP-Joint). For $\delta < \frac{1}{3}$ the only stationary contract that is WRP repeats in every period the stage game equilibrium (D,D). However, for $\delta = \frac{3}{10}$ there exists a WRP equilibrium that alternates between (C,C) and (D,C) on the equilibrium path.[12] The alternation makes the equilibrium path less attractive and thereby relaxes the incentives to renegotiate a punishment. An alternative possibility to relax the WRP conditions is to extend the game and stationary contracts by allowing players to burn money in payment stages. Skipping any details, we simply note that for $\delta = \frac{3}{10}$ there would then exist a WRP stationary contract with $a^e = (C,C)$ where each player burns $\frac{5}{12}$ units of money in every period on the equilibrium path.

This example suggests that Pareto optimal WRP equilibria may not always look plausible from an intuitive perspective on renegotiation-proofness. Since strong renegotiation-proofness is conceptually based on the Pareto-frontier of WRP equilibria, there is even no guarantee that SRP rules out effects like money burning on the equilibrium path. Despite this example, our characterization in Section 5.1 generally supports WRP and SRP as reasonable renegotiation-proofness concepts. As long as the discount factor is at least $\frac{1}{2}$, money burning on the equilibrium path will not facilitate renegotiation-proofness.

Furthermore, even if the discount factor is below $\frac{1}{2}$, we do not always find that lower values of $G(a^e)$ facilitate weak renegotiation-proofnes. This is only the case if condition (WRP-joint) is (alone) binding at the current discount factor. We now

[11] For a detailed characterization of WRP and SRP in general prisoner's dilemma games, see Section 5.3.

[12] Consider the paths

$$Q^0 = (p^0, (C,C), p^{DC}, (D,C), p^{CC}, (C,C), p^{DC}, (D,C), p^{CC}, ...)$$
$$Q^i_{pay} = (F^i, (C,C), p^{DC}, (D,C), p^{CC}, (C,C), p^{DC}, (D,C), p^{CC}, ...)$$
$$Q^i_{play} = (a^i, f^i, (C,C), p^{DC}, (D,C), p^{CC}, (C,C), p^{DC}, (D,C), p^{CC}, ...)$$

with $p^0 = p^{DC} = (0.6, -0.6)$, $p^{CC} = (0.823, -0.823)$, $a^1 = (C,D)$, $a^2 = (D,C)$, $F^i_i = \frac{1-\delta}{1-\delta^2}\left(1 - \delta p^{DC}_1 + \delta d - \delta^2 p^{CC}_1\right)$ and $f^i_i = F^i + \frac{g_i(a^i)}{\delta}$. Let σ be a simple strategy profile where play follows Q^0 whenever there was no unilateral deviation in the past and (re-)starts with Q^i_{pay} (Q^i_{play}) directly after any unilateral deviation of player i in play (pay) stage. By comparing continuation payoffs, it can be shown that σ constitutes a WRP equilibrium for $\delta = \frac{3}{10}$.

develop a helpful result that shows that if for a relevant subset of action plans the (WRP-joint) condition is not binding (or not alone binding) the characterization of Section 5.1 extends to smaller discount factors.

The relevant subset consists of action plans whose punishments are not dominated in the following sense:

Definition 7. We say an action plan (a^e, a^1, a^2) has WRP dominated punishments if there exist profiles $\tilde{a}^1$ and $\tilde{a}^2$ with $c_i(\tilde{a}^i) \leq c_i(a^i)$ and $G(\tilde{a}^i) \geq G(a^i)$ and one of the inequalities strict for at least one player $i = 1, 2$, such that there is a stationary contract with action plan $(a^e, \tilde{a}^1, \tilde{a}^2)$.

We now can formally state:

Proposition 10. *The results in Propositions* 7*,* 8*, and* 9 *also apply for a given* $\delta < \frac{1}{2}$*, if all action plans* (a^e, a^1, a^2) *without WRP dominated punishments that satisfy the conditions (SGP-*a^e*), (SGP-*a^i*), (WRP-i) also satisfy condition (WRP-Joint).*

Example 5.4 illustrates in detail how this result can be applied. While Proposition 8 is helpful to find WRP equilibria, the following result is helpful to rule out that certain subgame perfect equilibrium payoffs can be implemented in a weakly renegotiation-proof way.

Proposition 11. *There does not exist a WRP equilibrium with a joint equilibrium payoff* $\tilde{U}$*, if there exists no action plan* (a^e, a^1, a^2) *with* $G(a^e) \geq \tilde{U}$ *that fulfills conditions (SGP-*a^e*), (SGP-*a^i*), and (WRP-i).*

5.3 Example: Prisoner's Dilemma

In the first example, we investigate WRP and SRP for prisoner's dilemma games with the general payoff matrix from Section 3.2. First consider the case $S \leq 0$. In this case, the asymmetric profiles (C, D) and (D, C) are never used because they have a weakly lower cheating payoff and joint payoff than the Nash equilibrium. Using the simplified WRP constraint (14) for the symmetric case, we find that a stationary contract with action plan $(C, C), (D, D), (D, D)$ is WRP if and only if $\delta \geq \frac{1}{2}$. The reason for this result is that with maximal fines a punisher's continuation payoff before play stage is 2δ, which weakly exceeds the equilibrium path payoff of 1 if and only if $\delta \geq \frac{1}{2}$. The condition in Proposition 9 for strong renegotiation-proofness is also fulfilled. The only alternative WRP stationary contract with which we must compare the punisher's continuation payoffs is described by an infinite repetition of (D, D), which obviously yields a lower payoff. Hence, for the implementation of cooperative equilibrium play $a^e = (C, C)$, WRP and SRP requirements tighten the original subgame perfection condition to $\delta \geq \max\{\frac{1}{2}, \frac{d-1}{d}\}$. For smaller discount factors only an infinite repetition of the stage game Nash equilibrium is WRP and SRP.

Next consider the case $S > 0$. If only (C, C) and (D, D) are admissible, the situation is as in the case $S \leq 0$ above. In all cases in which (C, C) is not admissible, strong perfect equilibrium payoffs exist and are the same as SRP payoffs. The only remaining case is that all action profiles are admissible. Using Proposition 6, we find from (WRP-joint) that a stationary contract with action plan (C, C),(C, D),(D, C) can be WRP if and only if $\delta \geq \frac{1-S}{2-S}$. In case $\delta \geq \frac{1}{2}$ Propositions 7-9 can be applied and imply that the whole Pareto frontier of subgame perfect equilibrium payoffs is SRP.

If $\frac{1-S}{2-S} \leq \delta \leq \frac{1}{2}$, Proposition 10 can be used to obtain the same result. Basically, in this range (WRP-joint) is not yet binding and everything is fine. If $\frac{d-1}{d} < \delta < \frac{1-S}{2-S}$ and $d > 2 - S$, we still find that the sets of SRP and subgame perfect equilibrium payoffs coincide. Now this is an implication of our result for strong perfection. Yet, if $d \leq 2 - S$, our results do not allow a full characterization of SRP or even WRP payoffs for this range of discount factors. As seen in the example in Section 5.2, non-stationary equilibrium play or money burning may then be ways to increase WRP payoffs.

5.4 Example: Abreu's Simple Cournot Game

As next example, we consider Abreu's Cournot game from Section 3.1. We start by applying rather mechanical Proposition 10 to find the smallest discount factor above which the results of Proposition 7, 8 and 9 hold. The relevant calculations are illustrated in Table 1:

The first column shows a list of relevant action plans.[13] The second column shows the minimal discount factor for which a stationary contract with that action plan exists. If it is above $\frac{1}{2}$, these action plans are not relevant for Proposition 10 and can be ignored. The third column shows the minimal discount factor for which condition (WRP-joint) is satisfied. In all but two cases, this discount factor is smaller than the critical discount factor for subgame perfection, i.e. this action plans do not violate the condition in Proposition 10. For the two cases, we calculate the critical discount factor for the (WRP-i) conditions and find that it is also below the discount factor for (WRP-joint). Yet for the action plan $(M, M), (M, H), (H, M)$ the punishments are always WRP dominated by punishing with the Nash equilibrium (M, M) and

[13] Action plans for which no stationary contracts can exist are ommited, as well as action plans with $G(a^e) = G(a^1) = G(a^2)$ because they always satisfy (WRP-joint).

action plan	SGP	WRP-joint	WRP-1 und 2
$(L,L),(L,L),(H,L)$	$\delta \geq \frac{2}{3}$		
$(L,L),(L,M),(M,L)$	$\delta \geq \frac{5}{8}$		
$(L,L),(L,M),(H,L)$	$\delta \geq \frac{10}{23}$	$\delta \geq \frac{2}{15}$	
$(L,L),(L,H),(H,L)$	$\delta \geq \frac{1}{3}$	$\delta \geq \frac{3}{20}$	
$(M,M),(L,H),(H,L)$	$\delta \geq \frac{4}{11}$	$\delta \geq 0$	
$(M,M),(M,H),(H,M)$	$\delta \geq \frac{3}{10}$	$\delta \geq \frac{6}{13}$	$\delta \geq \frac{3}{10}$
$(L,M),(L,H),(H,L)$	$\delta \geq \frac{4}{13}$	$\delta \geq \frac{2}{11}$	
$(L,M),(M,H),(H,M)$	$\delta \geq \frac{1}{4}$	$\delta \geq \frac{8}{17}$	$\delta \geq \frac{7}{16}$

Table 1: Illustration of how to apply Proposition 10, by calculating critical discount factors of the different subgame perfection and WRP conditions are binding.

for the action plan $(L,M),(M,H),(H,M)$ the punishments are WRP dominated by (L,H) and (L,H) whenever $\delta \geq \frac{4}{13}$. This means, we can characterize optimal WRP and SRP equilibrium payoffs using stationary contracts if $\delta \geq \frac{4}{13}$.

First, we consider the case $\delta \geq \frac{1}{3}$. We start with testing whether full collusion with optimal penal codes can be weakly renegotiation-proof. From the optimal punishment profiles we choose the one that has the maximum joint payoff, that is, we look at the symmetric action plan $(L,L),(L,H),(H,L)$. We can use condition (14) to see whether this plan can be part of a WRP stationary contract. Since the condition equals $\delta \geq \frac{3}{13}$, we find that for $\delta \geq \frac{1}{3}$ the whole Pareto frontier $\mathcal{P}^{pay}_{SGP}$ is WRP. Next, we ask whether the stationary contracts with action plan $((L,L),(L,H),(H,L))$ are also SRP. To this end, we test the action plan against other WRP stationary contracts as required in Proposition 9. Since of all punishment profiles with the same cheating payoff only the one with the highest joint payoff matters, we only have to test against the action plan (L,L),$(L,M),(M,L)$. For this action plan the condition (15) equals $\delta \geq \frac{4}{11}$, hence the optimal WRP contract is also SRP.

Next, the case that $\delta < \frac{1}{3}$. Similar arguments show that with the same punishment profiles, $a^1 = (L,H)$ and $a^2 = (H,L)$, the partially collusive equilibrium play of $a^e = (M,L)$ is WRP and SRP for all $\delta \in [\frac{4}{13}, \frac{1}{3})$. For $\delta < \frac{4}{13}$, we apply Proposition 11 and find that no stationary contract with surplus strictly greater than 14 satisfies (WRP-i). Thus for $\delta < \frac{4}{13}$ infinite repetition of the stage game Nash equilibrium (M,M) is a SRP equilibrium.[14]

[14]Our conditions do not allow to rule out that there might exist other non-stationary SRP contracts with joint payoff 14 and perhaps a different distribution of joint payoffs than (7,7) if $\delta \in [\frac{3}{10}, \frac{4}{13})$. For $\delta < \frac{3}{10}$, it follows from Proposition 10 that the set of SRP payoffs is given

To summarize, we find that WRP and SRP restrict the set of possible punishment profiles, and for $\delta \in (\frac{1}{4}, \frac{4}{13})$ even reduce the maximal achievable equilibrium payoffs. Recall that collusive outcomes under subgame perfection are sustained for the largest set of discount factors if $a^1 = (M, H)$ and $a^2 = (H, M)$ are used as punishment. However, if weak renegotiation-proofness is required, the more efficient punishments $a^1 = (L, H)$ and $a^2 = (H, L)$ can sustain collusive play for a larger range of discount factors.

5.5 Example: Bertrand competition with symmetric costs

We now investigate the case of a Bertrand duopoly with monetary transfers. To have a compact strategy space and well defined cheating payoffs, we assume that prices a_i are chosen from a finite grid $M = \{m\varepsilon\}_{m=0}^{\overline{m}}$, where $\varepsilon > 0$ measures the grid size and $\overline{m}$ is a sufficiently large upper bound. Firm i's profits are given by

$$g_i(a) = \begin{cases} (a_i - k)D(a_i) & \text{if } a_i < a_j \\ (a_i - k)\frac{D(a_i)}{2} & \text{if } a_i = a_j \\ 0 & \text{if } a_i > a_j \end{cases},$$

where D is a weakly decreasing, nonnegative market demand function and $k \in M$ denotes the constant marginal costs that are identical for both firms. Clearly, marginal cost pricing is an optimal punishment for both firms. Furthermore, in every stationary contract that yields an equilibrium price between marginal cost and the monopoly price, it holds true that $c_i(a^e) = G(a^e) - \psi_i(\varepsilon)$, where $\psi_i(\varepsilon)$ is some nonnegative function that converges to 0 as $\varepsilon \to 0$.

For the limit $\varepsilon \to 0$, condition (SGP-a^e) implies that any such collusive price is sustainable if and only if $\delta \geq \frac{1}{2}$. (Note that it does not matter whether both firms supply the market equally or only one firm supplies the market and compensates the other firm.)

A discount factor of $\frac{1}{2}$ is also the minimal discount factor to sustain collusive prices as a subgame perfect equilibrium in a Bertrand duopoly without side payments, i.e. if only subgame perfection is considered this result may suggest that monetary transfers do not facilitate collusion. However, Lemma 6 implies that for all $\delta \geq \frac{1}{2}$ these collusive prices can also be sustained by a weakly renegotiation-proof stationary contract. Moreover, since $G(a^i)(1-\delta) - c_i(a^i)$ is maximized when $a_i^i \geq a_j^i = k$,[15] using Proposition 9, one can establish that the monopoly price can be sustained

$\{(7, 7)\}$.

[15]This can be seen by noting that first, the joint payoff is determined by the smaller of the two prices, and second, the deviating player can always choose a price just below this smaller price, hence $c_i(a^i) \geq G(a^i) + \psi_i(\epsilon)$.

even by a strongly renegotiation-proof stationary contract that uses maximal fines and marginal cost pricing as punishment.

For a Bertrand duopoly without monetary transfers, Farrell and Maskin (1989) establish that only marginal cost pricing can be sustained in a WRP equilibrium in pure strategies. Based on this result, McCutcheon (1997) argues that small fines on meetings where prices are discussed can facilitate collusion, since renegotiation becomes harder. Although for very large discount factors, collusive outcomes can be sustained as a WRP equilibrium if one allows for mixed strategies (Farrell and Maskin, 1989) or if prices must be choosen from a sufficiently coarse grid (Andersson and Wengström, 2007), the possibility for renegotiation-proof collusion for intermediate discount factors is generally reduced if WRP is required. Our example shows that with transfers, neither weak or strong renegotiation-proofness restricts the set of discount factors for which perfect collusion is possible. Thus, while the effect of meetings in smoke filled rooms on collusion may be ambiguous, this result makes clear that collusion is facilitated if participants of such meetings can easily swap briefcases filled with cash.

5.6 Example: Bertrand competition with asymmetric costs

Miklos-Thal (2011) shows that cost asymmetries facilitate the existence of collusive subgame perfect equilibria in repeated Bertrand competition if side payments are possible. We use our general characterization to replicate her results for a Bertrand duopoly and then show that weak renegotiation-proofness does not restrict the set of equilibrium payoffs.

There are two firms $i = 1, 2$ with constant marginal cost k_1 and k_2 with $k_1 < k_2$. We will characterize optimal subgame perfect and WRP contracts for the game considering the limit of continuous payments $\varepsilon \to 0$.

Let $\pi_1(a_1) = (a_1 - k_1)D(a_1)$ denote firm i's profits if it serves the whole market at a price a_1. As punishment profiles we choose $a^i = (k_i, k_i + \varepsilon)$, which guarantees a cheating payoff of $c_i(a^i) \approx 0$ to the punished firm. In the punishment of firm 2, firm 1 it gets a positive profit of $\pi_1(k_2)$, while firm 2 makes zero profits in the punishment of firm 1.

It follows from condition (SGP-a^e) that collusion is easiest to sustain if the low cost firm 1 supplies the whole market and compensates the high cost firm 2.[16] We consider equilibrium action profiles $a^e = (a_1^e, a_1^e + \varepsilon)$ where a_1^e is a price above firm 1's

[16] Joint payoffs G are maximized if firm 1 conducts the whole production. Since cheating payoffs result from marginally undercutting the equilibrium price, they do not depend on who serves the market (at least not in the limit of continuous prices $\varepsilon \to 0$).

marginal cost and weakly below firm 1's monopoly price. For small ε, corresponding cheating payoffs for firm 1 and 2 are $c_1(a^e) = \pi_1(a_1^e)$ and $c_2(a^e) \approx \phi(a_1^e)\pi_1(a_1^e)$ where $\phi(a_1^e) \equiv \frac{a_1^e - k_2}{a_1^e - k_1}$ is the ratio of firm 2's markup to firm 1's markup. Condition (SGP-a^e) thus implies that an equilibrium price a_1^e is sustainable if and only if

$$\delta \geq \frac{\phi(a_1^e)}{1 + \phi(a_1^e)}. \tag{16}$$

Since $\phi(a_1^e) < 1$, this critical discount factors is smaller $\frac{1}{2}$, which means cost asymmetries indeed facilitate collusion. Moreover, since ϕ is continuous and $\phi(k_2) = 0$ some collusive markup can be sustained for every discount factor $\delta > 0$.

Such contracts are also always weakly renegotiation-proof. The condition WRP-1 turns out to be directly equivalent to the subgame perfection condition (16). Condition (WRP-Joint) and condition (WRP-2) coincide and become

$$\delta \geq \frac{\pi_1(a_1^e) - \pi_1(k_2)}{2\pi_1 (a_1^e) - \pi_1(k_2)}. \tag{17}$$

For a completely inelastic demand function D, condition (17) is identical to the subgame perfection condition (16), and since D is weakly decreasing, condition (17) is weaker than condition (16).

6 Summary

We have shown that Pareto optimal subgame perfect payoffs and renegotiation-proof payoffs can generally be found by restricting attention to a simple class of stationary contracts. These stationary contracts prescribe play of the same action profile in every period on the equilibrium path, and punishments never last longer than one period, after which equilibrium play resumes. The first part of our paper establishes simple conditions that allow a quick characterization of Pareto-optimal subgame perfect equilibria with optimal penal codes for general two player stage games and side payments.

In the second part of the paper, we compared and characterized different concepts of renegotiation-proofness for intermediate discount factors. First we established that if renegotiation-proofness can take place only before the payment stage, every Pareto optimal subgame perfect payoff can be implemented in a renegotiation-proof way. But even if renegotiation is possible at all stages, transfers sometimes allow optimal payoffs to be strong perfect. We derived simple conditions to check whether given payoffs are robust against renegotiation in this sense. We also investigated the less restrictive concepts of weak and strong renegotiation-proofness. While in many examples, Pareto-optimal subgame perfect payoffs can be implemented as WRP or even SRP equilibria, this is not always the case: Pareto-optimal subgame perfect equilibria that rely on very inefficient punishments can fail to be renegotiation-proof. We also illustrated that in some cases optimal WRP equilibria can be obtained by artificially reducing equilibrium payoffs by alternating between an efficient and less efficient profile on the equilibrium path or by burning money on the equilibrium path. Such equilibria do not seem very plausible from an intuitive perspective of renegotiation-proofness. We have shown, though, that if $\delta \geq \frac{1}{2}$ or another sufficient condition holds, such contractual features will never be necessary to achieve optimal WRP or SRP payoffs.

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References

Abreu, Dilip, 1988. "On the Theory of Infinitely Repeated Games with Discounting," Econometrica, 56, 383-396.

Andersson, Ola and Erik Wengström, 2007. "A Note on Renegotiation in Repeated Bertrand Duopolies," Economics Letters, 95(3), 398-401.

Baker, George, Robert Gibbons, and Kevin J. Murphy, 2002. "Relational Contracts and the Theory of the Firm," The Quarterly Journal of Economics, 2, 39-84.

Baliga, Sandeep and Robert Evans, 2000. "Renegotiation in Repeated Games with Side-Payments," Games and Economic Behavior, 33, 159-176.

Bernheim, Douglas B. and Debraj Ray, 1989. "Collective Dynamic Consistency in Repeated Games," Games and Economic Behavior, 1(4), 295-326.

Blonski, Matthias and Giancarlo Spagnolo, 2007. "Relational Efficient Property Rights," Working paper.

Doornik, Katherine, 2006. "Relational contracting in partnerships," Journal of Economics & Management Strategy, 15(2), 517–548.

Fong, Yuk-fai and Jay Surti, 2009, "On the Optimal Degree of Cooperation in the Repeated Prisoner's Dilemma with Side Payments," Games and Economic Behavior, 67(1), 277-291.

Farrell, Joseph and Eric Maskin, 1989. "Renegotiation in Repeated Games," Games and Economic Behavior, 1(4), 327-360.

Goldlücke, Susanne and Sebastian Kranz, 2012, "Infinitely Repeated Games with Public Monitoring and Monetary Transfers", Journal of Economic Theory, 147(3), 1191-1221.

Harrington, Joseph E., 2006. "How do Cartels Operate?," Foundations and Trends in Microeconomics, 2(1).

Harrington, Joseph E. and Andrzej Skrzypacz, 2007 "Collusion with Monitoring of Sales," RAND Journal of Economics, 38(2), 314-331.

Levin, Jonathan, 2003. "Relational Incentive Contracts," American Economic Review, 93, 835-857.

Mailath, George J. & Obara, Ichiro & Sekiguchi, Tadashi, 2002. "The Maximum Efficient Equilibrium Payoff in the Repeated Prisoners' Dilemma," Games and Economic Behavior, 40(1), 99-122.

McCutcheon, Barbara, 1997. "Do Meetings in Smoke-Filled Rooms Facilitate Collusion?", Journal of Political Economy, 105(2), 330-350.

Miller David A. and Joel Watson, 2012. "A Theory of Disagreement in Repeated Games with Bargaining." Working paper.

Miklos-Thal, Jeanine, 2011. "Optimal Collusion under Cost Asymmetry," Economic Theory, 46(1), 99-125.

Sorin, Sylvain, 1986. "On Repeated Games with Complete Information", Mathematics of Operations Research, 11, 147-160.

Rayo, Luis, 2007. “Relational Incentives and Moral Hazard in Teams”, Review of Economic Studies. 74(3), 937-963.

Rubinstein, Ariel, 1980. "Strong perfect equilibrium in supergames", International Journal of Game Theory, 9(1), 1-12.

Appendix A: Strong Perfection and Non-Maximal Fines

This Appendix gives an example for the atypical case that only strongly perfect stationary equilibria exist in which a punished player does not pay maximum fines to the punisher. The stage game is described by the following payoff matrix

	A	B	D	E
A	25,-10	*,*	*,-5	*,*
B	*,*	-10,20	*,31	*,*
D	30,*	1,*	-1,-1	*,*
E	*,*	*,*	*,*	-5,-5

A * corresponds to a very low payoff, say -1000. The stage game has two Nash equilibria (D,D) and (E,E). The profile (E,E) is a harsher punishment as it gives lower payoffs to both players. The profiles (A,A) and (B,B) both yield higher joint payoffs than the two Nash equilibria. (A,A) differs from (B,B) by having higher joint payoffs (15 vs 10) and lower joint incentives to deviate (10 vs 22). Furthermore, (A,A) gives higher payoffs to player 1, while (B,B) gives higher payoffs to player 2.

We characterize strongly perfect equilibria for $\delta = \frac{1}{2}$. In this case an optimal subgame perfect stationary contract has the action plan $\bar{a}^e = (A, A)$ and $\bar{a}^1 = \bar{a}^2 = (E, E)$ and the admissible action profiles are (A,A),(B,B),(D,D) and (E,E). Figure 3 shows the payoffs of auxiliary contracts that play the corresponding action profile in the first period.

The Pareto frontier of subgame perfect continuation payoffs before play stage is given by the line segments of the auxiliary contracts for (B,B) and (A,A) and an interior part of the (D,D) line segment that is indicated by the two helper lines. The left point of the interior part on (D,D) is characterized by the highest payoff that can be given to player 1 in an auxilliary contract starting with (B,B). Using formula (9), we find that is given by

$$u_1^* = \frac{1}{2}(G(B,B) + G(A,A)) - \frac{1}{2}(c_2(B,B) + c_2(E,E)) = -\frac{1}{2}.$$

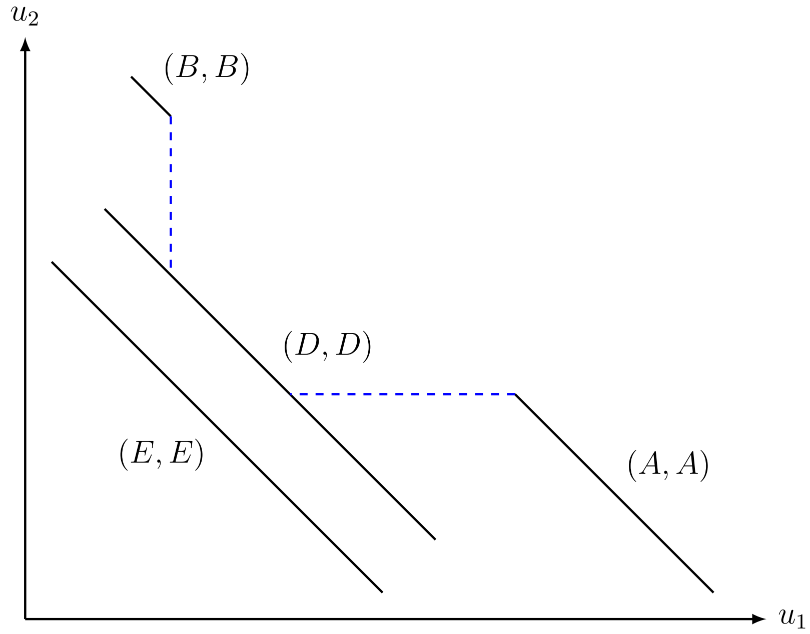

Figure 3: Payoffs of auxiliary contracts and Pareto frontier before play stage

Similarly, the right point on the interior part of (D,D) is characterized by the highest payoff that can be given to player 2 in an auxiliary contract starting with (A,A):

$$u_2^* = G(A,A) - \frac{1}{2}(c_1(A,A) + c_1(E,E)) = \frac{5}{2}.$$

It is straightforward to show that (A,A) can be implemented with a stationary equilibrium that uses (D,D) as punishment profile and non-maximal fines that yield punishment payoffs $u_1^* + \varepsilon$ and $u_2^* + \varepsilon$ for sufficiently small ε. Such a stationary equilibrium is strong perfect, while a stationary equilibrium with maximum fines is not.

For $\delta = \frac{1}{2}$, all strong perfect stationary equilibria require (D,D) as punishment profile for both players. One could alternatively think of punishing player 1 by playing (B,B), but one can verify that (B,B) can only be implemented if one uses (E,E) as punishment for player 2, but no equilibrium that uses (E,E) is strong perfect.

Appendix B: Proofs

<u>Proof of Lemma 1:</u> We are interested in finding conditions on a^e, a^1, a^2 that make it possible to define the equilibrium transfer p^e and fines F^1 and F^2 such that

conditions (6), (4), (5) for subgame perfection are fulfilled. Note that there are three conditions that bound u_i^i, $i = 1, 2$ from above, but only condition (4) bounds it from below. Therefore, these conditions hold for some u_i^i if and only if they hold for the lowest possible punishment payoffs $u_i^i = c_i(a^i)$, which are achieved by maximum fines $F_i^i = \frac{\delta u_i^e - c_i(a^e)}{1-\delta}$. With these maximum fines, condition (4) becomes (SGP-a^i).

Equilibrium transfers p^e then only appear in the conditions (6):

$$g_i(a^e) - \delta p_i^e \geq c_i(a^e)(1-\delta) + \delta c_i(a^i) \text{ for } i \in \{1, 2\}$$

Choosing $\delta p_1^e = g_1(a^e) - c_1(a^e)(1-\delta) - \delta c_1(a^1)$, these conditions bind exactly for player 1, and the condition for player 2 becomes condition (SGP-a^e).■

Proof of Proposition 1: Consider three sequences of subgame perfect equilibria

$$\{\sigma^e(n), \sigma^1(n), \sigma^2(n)\}_{n \in \mathbb{N}}$$

in Σ_{SGP}^{play} with $U(\sigma^e(n)) \to \overline{U}_{SGP}$ and $u_i(\sigma^i(n)) \to \overline{u}_{SGP}^i$ as $n \to \infty$. Let $a^k(n)$ be the first action profile on the equilibrium path of $\sigma^k(n)$ for $k \in \{e, 1, 2\}$. Then $a^k(n)$ is a sequence in the compact set A, and as such must have convergent subsequences with limits in A. We assume w.l.o.g. that these convergent subsequences are already given by $a^k(n)$ and denote their limits by $\bar{a}^e, \bar{a}^1$ and $\bar{a}^2$, respectively. In the following we use the properties of $\sigma^e(n), \sigma^1(n), \sigma^2(n)$ to make inferences about $\bar{a}^e, \bar{a}^1$ and $\bar{a}^2$. First, if we decompose $\sigma^e(n)$ into current and future payoff, we see that

$$U(\sigma^e(n)) \leq (1-\delta)G(a^e(n)) + \delta\overline{U}_{SGP}. \tag{18}$$

Here we used that $\bar{U}_{SGP} = sup_{u \in \mathcal{U}_{SGP}^{pay}} u_1 + u_2$. Because we look at the sum of payoffs, all payments cancel out. Since G is continuous, taking the limit $n \to \infty$ yields

$$\overline{U}_{SGP} \leq G(\bar{a}^e). \tag{19}$$

Second, subgame perfection of $\sigma^i(n)$ implies

$$u_i(\sigma^i(n)) \geq (1-\delta)c_i(a^i(n)) + \delta\overline{u}_{SGP}^i. \tag{20}$$

This condition holds because the payoff from staying on the path must be larger than cheating in the play stage, not making a transfer in the subsequent pay stage, and suffering the consequences, which cannot be worse than a payoff of $\overline{u}_{SGP}^i$.

Since c_i is continuous, taking the limit $n \to \infty$ yields

$$\overline{u}_{SGP}^i \geq c_i(\bar{a}^i). \tag{21}$$

Third, summing up the subgame perfection conditions of players 1 and 2 for $\sigma^e(n)$ yields

$$\overline{U}_{SGP} \geq (1-\delta)\left(c_1(a^e(n)) + c_2(a^e(n))\right) + \delta(\overline{u}^1_{SGP} + \overline{u}^2_{SGP}). \tag{22}$$

In the limit, and using (19) and (21), this becomes

$$G(\bar{a}^e) \geq (1-\delta)\left(c_1(\bar{a}^e) + c_2(\bar{a}^e)\right) + \delta(c_1(\bar{a}^1) + c_2(\bar{a}^2)). \tag{23}$$

Last, we exploit the subgame perfection condition

$$u_j(\sigma^i(n)) \geq c_j(a^i(n))(1-\delta) + \delta\overline{u}^j_{SGP} \tag{24}$$

as well as

$$G(a^i(n))(1-\delta) + \delta\overline{U}_{SGP} \geq U(\sigma^i(n))$$

to get

$$G(a^i(n))(1-\delta) + \delta\overline{U}_{SGP} - \overline{u}^i_{SGP} \geq c_j(a^i(n))(1-\delta) + \delta\overline{u}^j_{SGP}. \tag{25}$$

In the limit, and using (19) and (21), this becomes

$$G(\bar{a}^i)(1-\delta) + \delta G(\bar{a}^e) - c_i(\bar{a}^i) \geq c_j(\bar{a}^i)(1-\delta) + \delta c_j(\bar{a}^j). \tag{26}$$

Equations (23) and (26) together with Lemma 1 now tell us that there is a stationary contract with action plan $(\bar{a}^e, a^1, a^2)$, with joint payoff $G(\bar{a}^e) = \overline{U}_{SGP}$, and punishment payoffs $c_i(\bar{a}^i) = \overline{u}^i_{SGP}$, i.e. $\bar{a}^e$ is indeed an optimal action profile and $\bar{a}^i$ indeed are optimal punishments. The regularity conditions that we imposed on a stationary contract are also fulfilled if in case that $c_i(\bar{a}^i) = c_i(\bar{a}^e)$ we replace $\bar{a}^i$ by $\bar{a}^e$. Different up-front payments can be used to achieve all payoffs on the line between $(c_1(\bar{a}^1), G(\bar{a}^e) - c_1(\bar{a}^1))$ and $(G(\bar{a}^e) - c_2(\bar{a}^2), c_2(\bar{a}^2))$. Since player i will never get a lower payoff than $c_i(\bar{a}^i) = \overline{u}^i_{SGP}$ in any subgame perfect equilibrium, this line constitutes the Pareto frontier of subgame perfect payoffs. ■

Proof of Proposition 2: First, we show that if an action profile relaxes a condition at some point in the algorithm, then it also relaxes this condition at all earlier rounds of the algorithm, and that as the algorithm continues, the equilibrium action profiles $a^e(n)$ can only become less efficient and the punishment action profiles $a^i(n)$ can only become weaker punishments. That is, we show that for all n with $1 \leq n \leq n^*$ the following two claims hold:

1. $G(a^e(n)) \leq G(a^e(n-1))$ and $c_i(a^i(n)) \geq c_i(a^i(n-1))$ for $i \in \{1, 2\}$.
2. $R^k(n) \subset R^k(n-1)$ for all $k \in \{e, 1, 2\}$

We first that show that claim 2 follows from claim 1. If it is the case that $R^k(n) = \varnothing$, the claim obviously holds. We therefore assume now that these sets are nonempty. Let $a^e \in R^e(n)$, i.e. a^e relaxes condition (SGP-a^e) of round n. This means that condition (SGP-a^e) with $a^e, a^1 = a^1(n)$ and $a^2 = a^2(n)$ holds for some $\tilde{\delta} < \delta^*(n)$. It must then also hold for $a^1 = a^1(n-1)$ and $a^2 = a^2(n-1)$, because their cheating payoffs are weakly smaller. Since $\tilde{\delta} < \delta^*(n-1)$, this means that $a^e \in R^e(n-1)$. Similarly, for $a^i \in R^i(n)$ condition (SGP-a^i) holds for $a^j = a^j(n)$ and $a^e = a^e(n)$ and some $\tilde{\delta} < \delta^*(n)$. It must then also hold for $a^j = a^j(n-1)$ and $a^e = a^e(n-1)$ since $c_j(a^j(n-1)) \leq c_j(a^j(n))$ and $G(a^e(n)) \leq G(a^e(n-1))$. That $R^i(n) \subset R^i(n-1)$ follows again because $\delta^*(n) \leq \delta^*(n-1)$.

Next, we show claim 1 by induction. For $n = 1$, it is obviously true. We now assume that it is true for n, which also implies that claim 2 holds for n, and show that it holds for $n+1$ as well. For $a^k(n+1) = a^k(n)$ the corresponding payoffs do not change. Assume $a^k(n+1) \neq a^k(n)$. Since $a^k(n+1) \in R^k(n)$, which by induction hypothesis is a subset of $R^k(n-1)$, we have $a^k(n+1) \in R^k(n-1)$ and for $k = e$

$$G(a^e(n)) = \max_{a^e \in R^e(n-1)} G(a^e) \geq G(a^e(n+1)),$$

while for $k = i \in \{1, 2\}$

$$c_i(a^i(n)) = \min_{a^i \in R^i(n-1)} c_i(a^i) \leq c_i(a^i(n+1)).$$

That is, claim 1 holds because one optimizes over a smaller set.

Note that at least one of the inclusions in claim 2 must hold strictly, because in every round of the procedure at least one of the candidate action profiles is replaced: If an action profile $a^k(n)$ is replaced, i.e. $\delta^k(n) \geq \delta^*(n)$, then we have $a^k(n) \in R^k(n-1)$ and $a^k(n) \notin R^k(n)$. Because we assumed a finite action space, this tells us that eventually $R^k(n) = \varnothing$ and the procedure terminates.

Finally, to prove that the procedure finds the right action plan for all discount factors, note first that the action plan $a^e(0), a^1(0), a^2(0)$ is clearly optimal for the range of discount factors $[\delta(0), 1)$ with $\delta(0) < 1$. Next, assume that an action plan $\bar{a}^e$,$\bar{a}^1, \bar{a}^2$ is optimal for a discount factor $\bar{\delta} < \delta(0)$. Let $\bar{n}$ be the round such that $\delta^*(\bar{n}) > \bar{\delta} \geq \delta^*(\bar{n}+1)$. We will show that $\bar{a}^k \in R^k(n)$ for all $k = e, 1, 2$ and $n \leq \bar{n}$, which implies that an optimal stationary contract is chosen for discount factor $\bar{\delta}$. Because SGP-a^e holds for $\bar{a}^e, \bar{a}^1, \bar{a}^2$ and $\bar{\delta} < \delta^*(n)$ for $n \leq \bar{n}$, the action profile $\bar{a}^e$ relaxes condition SGP-a^e in round n if $c_1(a^1(n)) + c_2(a^2(n)) \leq c_1(\bar{a}^1) + c_2(\bar{a}^2)$. Similarly, $\bar{a}^i$ relaxes condition SGP-a^i in round $n \leq \bar{n}$ if $c_j(a^j(n)) \leq c_j(\bar{a}^j)$ and $G(a^e(n)) \geq G(\bar{a}^e)$. It follows that $\bar{a}^k \in R^k(0)$ for all $k \in \{e, 1, 2\}$. To prove our claim, it only remains to show that if $\bar{a}^k$ relaxes condition SGP-a^k in round $n-1$, then it also relaxes the condition in round $n \leq \bar{n}$. This holds because $\bar{a}^e \in R^e(n-1)$ implies that $G(\bar{a}^e) \leq G(a^e(n))$, and $\bar{a}^i \in R^i(n-1)$ implies that $c_i(a^i(n)) \leq c_i(\bar{a}^i)$. As long as $n \leq \bar{n}$, these conditions together imply that $\bar{a}^k \in R^k(n)$ for all k.

For $\delta = 0$ there is always a stationary equilibrium with an optimal action plan consisting only of stage game Nash equilibria. The optimal action plan requires that a^e is the stage game Nash equilibrium with the highest joint payoff $G(a^e)$ and a^i is the stage game Nash equilibrium with the lowest cheating payoff $c_i(a^i)$ and it is straightforward that (SGP-a^e) and (SGP-a^i) are then satisfied. These stage game Nash equilibria will be elements of $R^k(n)$ for $n < n^*$. The algorithm thus terminates with $\delta(n^*) = 0$ and all $a^k(n^*)$ being stage game Nash equilibria.■

Proof of Proposition 3: Assume that a strong perfect equilibrium exists. For both players $i = 1, 2$, let $\bar{u}^i$ be a tuple in the closure of $\mathcal{U}_{SP}^{play}$ with $\bar{u}_i^i = \overline{u}_{SP}^i$. Since punishments with continuation payoffs $\bar{u}_i^i$ must be able to sustain at least one optimal action profile $\bar{a}^e$, it must hold that

$$G(\bar{a}^e) \geq (c_1(\bar{a}^e) + c_2(\bar{a}^e))(1-\delta) + \delta(\bar{u}_1^1 + \bar{u}_2^2). \tag{27}$$

By similar steps as in the proof of Proposition 1, we find that for both player $i = 1, 2$ there must exist $a^i \in A$ with $\bar{u}_i^i \geq c_i(a^i)$ and

$$G(a^i)(1-\delta) + \delta G(\bar{a}^e) - \bar{u}_i^i \geq \bar{u}_j^i \geq c_j(a^i)(1-\delta) + \delta \bar{u}_j^j. \tag{28}$$

Conditions (27) and (28) imply that there must exist a stationary contract with action plan $(\bar{a}^e, a^1, a^2)$ and punishment payoffs $\bar{u}_1^1$ and $\bar{u}_2^2$. In this stationary contract, all continuation equilibria (at payment or play stage) either have total payoff $\overline{U}_{SGP}$, or a continuation payoff of u^i with $u_i^i = \bar{u}_i^i$ and $u_j^i \geq \bar{u}_j^i$ (the latter follows from condition (28)).■

Proof of Lemma 2: Player i has no incentive to deviate from $\tilde{a}$ if and only if the payments at the beginning of the second period satisfy

$$g_i(\tilde{a}) - \delta \tilde{p}_i \geq c_i(\tilde{a}) - \delta F_i. \tag{29}$$

where $F_i = \frac{\delta}{1-\delta}(u_i^e - c_i(a^i))$ are the maximum fines. If that constraint is satisfied, player i will also be willing to pay $\tilde{p}_i$ in the next period, since he is willing to pay F_i and we have $c_i(\tilde{a}) \geq g_i(\tilde{a})$. Given a payment $\tilde{p}_i$, player i's expected utility before the first play stage is

$$\tilde{u}_i = (1-\delta)(g_i(\tilde{a}) - \delta \tilde{p}_i) + \delta u_i^e \tag{30}$$

Solving (30) for $\tilde{p}_i$, substituting into (29) and rearranging yields

$$\tilde{u}_i \geq \delta c_i(a^i) + (1-\delta) c_i(\tilde{a}).$$

This means those and only those distributions of the joint continuation payoff $(1-\delta)G(\tilde{a})+\delta G(\bar{a}^e)$ that give every player at least $\delta c_i(a^i)+(1-\delta)c_i(\tilde{a})$ can be implemented by an auxiliary contract with first period action profile $\tilde{a}$. We also find that an auxiliary contract exists if and only if $\sum_{i=1}^{2}(\delta c_i(a^i)+(1-\delta)c_i(\tilde{a})) \leq (1-\delta)G(\tilde{a})+\delta G(\bar{a}^e)$, which is equivalent to condition (8).■

Proof of Proposition 4: Consider any subgame perfect continuation equilibrium $\tilde{\sigma}$ at play stage that starts with $\tilde{a}$ as the first action profile. Since $c_i(\bar{a}^i)$ is the lowest subgame perfect payoff for player i, each player i must get at least a continuation payoff of $\delta c_i(\bar{a}^i)+(1-\delta)c_i(\tilde{a})$. Furthermore, the joint continuation payoff cannot exceed $(1-\delta)G(\tilde{a})+\delta G(\bar{a}^e)$. But this implies that $\tilde{a}$ is admissible and Lemma (2) guarantees that there is an auxiliary contract with payoffs that are equal to or Pareto-dominate the payoffs of $\tilde{\sigma}$.■

Proof of Proposition 5: First, we show that a stationary contract σ with action plan $(\bar{a}^e, a^1, a^2)$ and punishment payoffs u_1^1 and u_2^2 is strong perfect given that the conditions listed in the proposition hold. Clearly, continuation equilibria that start in the payment stage or before play of $\overline{a}^e$ cannot be Pareto-dominated. We only have to show that none of the continuation equilibria in which a player is punished in play stage is Pareto-dominated. Assume to the contrary that there exists a continuation equilibrium $\tilde{\sigma} \in \Sigma_{SGP}^{play}$ that strictly Pareto-dominates the punishment for player i. The first action $\tilde{a}$ of $\tilde{\sigma}$ is admissible and since

$$G(a^i)(1-\delta)+\delta G(\bar{a}^e) < U(\tilde{\sigma}) \leq G(\tilde{a})(1-\delta)+\delta G(\bar{a}^e)$$

it must hold that $G(\tilde{a}) > G(a^i)$, hence either inequality (SP1) or (SP2) holds. In the equilibrium $\tilde{\sigma}$ player j's payoff is bounded above by the joint payoff $U(\tilde{\sigma})$ minus player i's minimum payoff $(1-\delta)c_i(\tilde{a})+\delta\overline{u}_{SGP}^i$. Hence, Pareto-dominance of $\tilde{\sigma}$ implies that

$$(1-\delta)G(a^i)+\delta G(\bar{a}^e)-u_i^i < u_j(\tilde{\sigma}) \leq (1-\delta)G(\tilde{a})+\delta G(\bar{a}^e)-(1-\delta)c_i(\tilde{a})-\delta\overline{u}_{SGP}^i$$

and

$$u_i^i < u_i(\tilde{\sigma}) \leq G(\tilde{a})(1-\delta)+\delta G(\bar{a}^e)-(1-\delta)c_j(\tilde{a})-\delta\overline{u}_{SGP}^j,$$

which leads to a contradiction to the fact that either (SP1) or (SP2) has to hold.

Next we assume that σ is a strong perfect stationary contract with action plan $(\bar{a}^e, a^1, a^2)$ and punishment payoffs u_1^1 and u_2^2, and show that the conditions stated in the proposition have to hold. Assume to the contrary that there exists an admissible action profile $\tilde{a}$ with $G(\tilde{a}) > G(a^i)$ and

$$(1-\delta)G(\tilde{a})+\delta G(\bar{a}^e)-(1-\delta)c_i(\tilde{a})-\delta\overline{u}_{SGP}^i > (1-\delta)G(a^i)+\delta G(\bar{a}^e)-u_i^i \quad (31)$$

as well as

$$(1-\delta)G(\tilde{a})+\delta G(\bar{a}^e)-(1-\delta)c_j(\tilde{a})-\delta\overline{u}_{SGP}^j > u_i^i. \quad (32)$$

for some player i. Because $\tilde{a}$ is admissible, there exists an auxiliary stationary contract in which each player gets strictly more than in the punishment phase of σ. It follows that σ is not strong perfect.■

Proof of Corollary 2: If for all admissible action profiles $\tilde{a}$ with $G(\tilde{a}) > G(\bar{a}^i)$ and both players $i = 1, 2$ condition (10) holds, then also condition (SP1) of Proposition 5 is true.■

Proof of Corollary 3: Assume to the contrary that there exists a strong perfect stationary contract with equilibrium action profile $\bar{a}^e$ and punishment profiles $a^i = a$ and a^j, with punishment payoffs u_1^1 and u_2^2. Take $\tilde{a}$ as in the corollary. Then because $G(\tilde{a}) > G(a^i)$, according to Proposition 5 one of the conditions SP1 and SP2 must hold. However, because $c_i(\bar{a}^i) \leq c_i(a^i) \leq u_i^i$, condition SP1 implies $G(a^i) - c_i(a^i) \geq G(\tilde{a}) - c_i(\tilde{a})$, which contradicts (11) for player i. And because of subgame perfection (condition SGP-a^i), condition SP2 implies

$$(1-\delta)(G(a^i) - c_j(a^i)) + \delta G(\bar{a}^e) - \delta u_j^j \geq (1-\delta)(G(\tilde{a}) - c_j(\tilde{a})) + \delta G(\bar{a}^e) - \delta c_j(\bar{a}^j),$$

which in turn implies $G(a^i) - c_j(a^i) \geq G(\tilde{a}) - c_j(\tilde{a})$, thereby contradicting condition (11) for the other player. ■

Proof of Corollary 4: For a proof by contradiction, assume that there is a strong perfect stationary contract with action plan $(\bar{a}^e, a^1, a^2)$ and punishment payoffs u_1^1 and u_2^2 such that for both players $i = 1, 2$ condition (11) holds and $G(\bar{a}^e) > G(a^i)$. We consider Proposition 5 for $\tilde{a} = \bar{a}^e$. Condition (SP1) does not hold for any player $i = 1, 2$. Therefore, condition (SP2) must hold for both players, and in sum these conditions imply that $u_1^1 + u_2^2 \geq G(\bar{a}^e)$. This can only be fulfilled if $u_1^1 + u_2^2 = G(\bar{a}^e)$, and in this case the condition also implies that $G(\bar{a}^e) = c_1(\bar{a}^1) + c_2(\bar{a}^2)$. This means that $\bar{a}^e = \bar{a}^1 = \bar{a}^2$ (due to the regularity condition we imposed on stationary contracts), which contradicts subgame perfection of the stationary contract with $G(\bar{a}^e) > G(a^i)$.■

Proof of Lemma 3: Assume that there exists a WRP stationary contract with action plan (a^e, a^1, a^2) and equilibrium payoffs u^e. If $(1-\delta)G(a^i) + \delta G(a^e) - c_i(a^i) < u_j^e$, then the stationary contract can only be WRP if must hold that $c_i(a^i) \geq u_i^e$, i.e., $c_i(a^i) \geq c_i(a^e)$. This implies $a^i = a^e$ and therefore $G(a^i) + \delta G(a^e) - c_i(a^i) = G(a^e) - c_i(a^e) \geq G(a^e) - u_i^e = u_j^e$.

Next, assume that for an action plan (a^e, a^1, a^2) and equilibrium payoff u^e inequality (13) holds. Since $G(a^e) \geq G(a^i)$ this implies that the payoff when player i is punished and the equilibrium payoff u^e cannot be Pareto-ranked. Moreover, $(1-\delta)G(a^i) + \delta G(a^e) - c_i(a^i) \geq u_j^e \geq c_j(a^j)$, and therefore the two punishments cannot be Pareto-ranked, either.■

Proof of Proposition 6: Conditions (WRP-i) and (WRP-Joint) follow from condition 13 and subgame perfection. For the other direction, assume there exists a stationary contract with action plan (a^e, a^1, a^2), which fulfills (WRP-i) and (WRP-Joint). These conditions and the subgame perfection conditions (see Lemma 1) imply that there exist net-payments p^e such that

$$G(a^e) - (1-\delta)c_2(a^e) - \delta c_2(a^2) \geq g_1(a^e) - \delta p_1^e \geq c_1(a^e)(1-\delta) + \delta c_1(a^1),$$

$$(1-\delta)G(a^2) + \delta G(a^e) - c_2(a^2) \geq g_1(a^e) - \delta p_1^e \geq (1-\delta)(G(a^e) - G(a^1)) + c_1(a^1).$$

Using these inequalities and Lemma 3, it is straightforward to verify that a stationary contract with action plan (a^e, a^1, a^2), maximal fines, and equilibrium payments p^e exists and is WRP.■

Proof of Proposition 7: Let σ be any WRP equilibrium and let

$$\bar{U} = \sup_{u \in \mathcal{U}^{play}(\sigma)} u_1 + u_2,$$

and

$$\bar{u}_i^i = \inf_{u \in \mathcal{U}^{play}(\sigma)} u_i.$$

We take $(\bar{u}_1^e, \bar{u}_2^e)$ to be a payoff tuple in the closure of $\mathcal{U}^{play}(\sigma)$ such that $\bar{u}_1^e + \bar{u}_2^e = \bar{U}$. Similarly, $(\bar{u}_1^i, \bar{u}_2^i)$ shall be a tuple in the closure of $\mathcal{U}^{play}(\sigma)$ such that among all such tuples that have the payoff $\bar{u}_i^i$ for player i, player j's payoff is maximized. We then have that $\bar{u}_i^i \leq u_i$ and $\bar{u}_j^i \geq u_j$ for all $u \in \mathcal{U}^{play}(\sigma)$. Let $u(\sigma|h^e(n))$ be a sequence in $\mathcal{U}^{play}(\sigma)$ with limit $(\bar{u}_1^e, \bar{u}_2^e)$ and for $i = 1, 2$ let $u(\sigma|h^i(n))$ be a sequence with limit $(\bar{u}_1^i, \bar{u}_2^i)$. Let furthermore $a^k(n)$ be the w.l.o.g. convergent sequences of the first action profiles of the continuation equilibria $\sigma|h^k(n)$, $k \in \{e, 1, 2\}$. Completely analogous to the proof of Proposition 1 we have for the limits of these sequences, denoted by a^e, a^1, a^2, that

$$G(a^e) \geq \bar{U},$$

$$c_i(a^i) \leq \bar{u}_i^i,$$

$$\bar{U} \geq (c_1(a^e) + c_2(a^e))(1-\delta) + \delta(\bar{u}_1^1 + \bar{u}_2^2),$$

$$G(a^i)(1-\delta) + \delta\bar{U} - \bar{u}_i^i \geq \bar{u}_j^i \geq c_j(a^i)(1-\delta) + \delta\bar{u}_j^j,$$

as well as

$$G(a^i)(1-\delta) + \delta\bar{U} - c_i(a^i) \geq \bar{u}_j^e \geq c_j(a^e)(1-\delta) + \delta c_j(a^j),$$

which also implies

$$(G(a^1) + G(a^2))(1-\delta) + 2\delta\bar{U} - c_1(a^1) - c_2(a^2) \geq \bar{U}.$$

Since we assumed that $\delta \geq \frac{1}{2}$, these conditions are relaxed if we replace $\bar{U}$ by $G(a^e)$. Next, we define $\tilde{a}^e \in \{a^e, a^1, a^2\}$ such that $G(\tilde{a}^e) = \max\{G(a^e), G(a^1), G(a^2)\}$, and we define $\tilde{a}^i = a^i$ if $c_i(a^i) < c_i(\tilde{a}^e)$ and $\tilde{a}^i = \tilde{a}^e$ else. By making these definitions, we obtain an equilibrium action profile with higher total payoff and punishment action profiles with lower cheating payoffs compared to the original action plan. Therefore, all conditions still hold for the new action plan $(\tilde{a}^e, \tilde{a}^1, \tilde{a}^2)$:

$$G(\tilde{a}^e) \geq (1-\delta)(c_1(\tilde{a}^e) + c_2(\tilde{a}^e)) + \delta(c_1(\tilde{a}^1) + c_2(\tilde{a}^2)),$$

$$G(\tilde{a}^i)(1-\delta) + \delta G(\tilde{a}^e) - c_i(\tilde{a}^i) \geq \max(c_j(\tilde{a}^e), c_j(\tilde{a}^i))(1-\delta) + \delta c_j(\tilde{a}^j),$$

and

$$(G(\tilde{a}^1) + G(\tilde{a}^2))(1-\delta) + 2\delta G(\tilde{a}^e) - (c_1(\tilde{a}^1) + c_2(\tilde{a}^2)) \geq G(\tilde{a}^e).$$

Because of Lemma 6 there is a WRP stationary contract with action plan $\tilde{a}^e, \tilde{a}^1, \tilde{a}^2$, which satisfies $G(\tilde{a}^e) \geq \bar{U}$, $c_i(\tilde{a}^i) \leq \bar{u}_i^i$, and $G(\tilde{a}^i)(1-\delta) + \delta G(\tilde{a}^e) - c_i(\tilde{a}^i) \geq \bar{u}_j^i$. It follows that for any WRP payoff $u(\sigma), \sigma \in \Sigma_{WRP}^{pay}$ there is a stationary contract that weakly Pareto dominates it, which then implies the proposition.■

Proof of Proposition 8: Since no payoff in $\mathcal{U}_{SRP}^{play}$ Pareto dominates the other, one can show as in the WRP case that there exists an action plan (a^e, a^1, a^2) such that $G(a^e) \geq u_1 + u_2$, and $G(a^i)(1-\delta) + \delta G(a^e) - u_{SRP}^i \geq u_j$ for all $u \in \mathcal{U}_{SRP}^{play}$, as well as $c_i(a^i) \leq u_{SRP}^i$,

$$G(a^e) \geq (c_1(a^e) + c_2(a^e))(1-\delta) + \delta(\bar{u}_{SRP}^1 + \bar{u}_{SRP}^2),$$

and

$$G(a^i)(1-\delta) + \delta G(a^e) \geq (c_j(a^i) + c_i(a^i))(1-\delta) + \delta(\bar{u}_{SRP}^1 + \bar{u}_{SRP}^2).$$

This tells us that there is a WRP stationary contract with action plan (a^e, a^1, a^2), and because it cannot Pareto dominate the SRP equilibria we have

$$G(a^e) = \max_{u \in \mathcal{U}_{SRP}^{play}} \{u_1 + u_2\}.$$

Because the worst SRP payoffs are able to sustain a^e it follows that there is a SRP stationary contract with action plan (a^e, a^1, a^2) and punishment payoffs u_{SRP}^i. ■

Proof of Proposition 9: Assume that σ is not SRP. Since σ is an optimal WRP stationary contract, it can only be dominated in the punishment phase, that is, there must be an $i \in \{1, 2\}$ and a WRP payoff $u \in \mathcal{U}_{WRP}^{play}$ such that $u_i > c_i(a^i)$ and

$$u_j > G(a^i)(1-\delta) + \delta G(a^e) - c_i(a^i).$$

It follows from the proof of Proposition 7 that there exists a WRP stationary contract with action plan $(\tilde{a}^e, \tilde{a}^1, \tilde{a}^2)$ that fulfills

$$G(\tilde{a}^i)(1-\delta) + \delta G(\tilde{a}^e) - c_i(\tilde{a}^i) \geq u_j.$$

■

Proof of Proposition 10: Note first that Propositions 8 and 9 assume that $\delta \geq \frac{1}{2}$ only because they rely on Proposition 7. The proof of Proposition 7 uses $\delta \geq \frac{1}{2}$ to show that the WRP-joint condition holds once it has established existence of an action plan (a^e, a^1, a^2) with the right properties that satisfies (SGP-a^e), (SGP-a^i) and (WRP-i). Let $\mathcal{A}(a^e)$ be the set of all punishment profiles a^1, a^2 that together with a^e satisfy the three conditions. Then let

$$(\tilde{a}^1, \tilde{a}^2) \in \arg \max_{a^1, a^2 \in \mathcal{A}(a^e)} G(a^1) + G(a^2) - c_1(a^1) - c_2(a^2).$$

With this definition, there can be no $a^i \in A$ with $c_i(a^i) \leq c_i(\tilde{a}^i)$ and $G(a^i) > G(\tilde{a}^i)$ such that the pair of action profiles a^i and $\tilde{a}^j$ is in $\mathcal{A}(a^e)$. It follows that the action plan (a^e,$\tilde{a}^1$,$\tilde{a}^2$) also satisfies (WRP-joint), and Proposition 7 holds.■

Proof of Proposition 11: It directly follows from the proof of Proposition 7 that if there exists a WRP equilibrium with a joint equilibrium payoff of $\tilde{U}$, then there exists an action plan (a^e, a^1, a^2) with $G(a^e) \geq \tilde{U}$ that fulfills conditions (SGP-a^e), (SGP-a^i), (WRP-i) and

$$(G(a^1) + G(a^2))(1 - \delta) + 2\delta\tilde{U} - c_1(a^1) - c_2(a^2) \geq \tilde{U}.$$

■