Investments as Signals of Outside Options

Susanne Goldlücke* and Patrick W. Schmitz†

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Abstract

Consider a seller who can make an observable but non-contractible investment to improve an intermediate good that is specialized to a particular buyer's needs. The buyer then makes a take-it-or-leave-it offer to the seller. The seller has private information about the fraction of the ex post surplus that he can realize on his own. Compared to a situation with complete information, additional investment incentives are generated by the seller's desire to pretend a strong outside option. On the other hand, ex post efficiency is not attained since asymmetric information at the bargaining stage sometimes leads to inefficient separations.

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*Corresponding author, Phone:+496211813072. University of Mannheim, L7,3-5, 68161 Mannheim, Germany. Email: goldluecke@uni-mannheim.de.

†University of Cologne, Albertus Magnus Platz, 50923 Cologne, Germany, and CEPR, London, UK. Email: patrick.schmitz@uni-koeln.de.

I Introduction

This paper offers a new perspective on the hold-up problem, which is a central ingredient of the modern property rights approach to the theory of the firm based on incomplete contracting. In the seminal contributions of Grossman and Hart (1986) and Hart and Moore (1990), an agent can make an observable but non-contractible investment that increases the surplus that can be generated within a given relationship more than it increases the agent's default payoff (i.e., the payoff that he can realize outside of the relationship).[1] When the investing party does not have all the bargaining power ex post, it does not get the full returns of its investment, so that in general there is an underinvestment problem. The fact that investments are partly (but not fully) relationship-specific is crucial in this literature, because all that governance structures (e.g., ownership arrangements) affect is what a party can get outside of the relationship. It is a standard assumption that there is symmetric information between the parties, so that they always agree on the ex post efficient decision to collaborate, but ex ante investment incentives depend on the payoffs that the parties could achieve outside of the relationship, so that institutions matter.

More recently, several authors have argued that the incomplete contracting literature may have overemphasized the relevance of encouraging ex ante investments while it has almost completely neglected the possibility of ex post inefficiencies. In particular, Williamson (2000, p. 605) emphasizes that this is the "most consequential difference" between transaction cost economics and the property rights theory.[2] In this paper, we take up this line of criticism, by assuming that a party may have better information than its trading partner about the fraction of the surplus that the party can realize

[1]For a recent survey of the literature, see Segal and Whinston (2013), who point out that "*hold-up* models, whose use for examining the optimal allocation of property rights began with the seminal contribution of Grossman and Hart (1986), have been a workhorse of much of organizational economics over the last 20 years" (p. 103). See also Hart (1995) for a comprehensive exposition.

[2]Williamson (2002, p. 188) argues it is "deeply problematic" that the incomplete contracting models assume ex post efficient bargaining under symmetric information. Holmström and Roberts (1998) and Whinston (2003) also point out that the standard property rights models might be too narrowly focused on the underinvestment problem.

on its own.[3] Under this plausible assumption, underinvestment problems are ameliorated and ex post inefficiencies become relevant; i.e., the incomplete contracting approach moves closer to transaction cost economics in the sense of Williamson (1975, 1985).

Specifically, consider a seller who can invest in order to increase the value of an intermediate good. The good is specialized to the needs of a particular buyer. The parties cannot write a contract ex ante. If the parties do not reach an agreement ex post, the seller can realize only a fraction $\theta \leq 1$ of the ex post surplus on his own. Hence, it is always ex post efficient for the two parties to trade the intermediate good. For simplicity, we assume that the buyer can make a take-it-or-leave-it offer ex post, so that the hold-up problem is most severe. Under complete information, ex post efficiency would always be achieved, but the seller would underinvest, since the buyer would hold up the seller; i.e., she would offer only a fraction θ of the gains from trade.

Our key innovation is to assume that from the outset the seller has private information about the fraction θ of the ex post surplus that he can realize on his own.[4] It turns out that the seller's private information may stimulate larger investment levels compared to the case of complete information, because there is a signaling motive in the seller's investment choice. The buyer will try to deduce the seller's outside option from the chosen level of investment. If the seller chooses a small investment level, it seems likely that he has a weak outside option, so that the buyer will then indeed make a low offer. If instead the seller chooses a large investment level, the buyer may believe that the seller has a strong outside option, in which case she would have to make a high offer. Hence, a seller with a weak outside option may have an incentive to mimic a seller with a strong outside option. It turns out that

[3] Our contribution is thus in line with Holmström (1999), who points out that the assumption in the incomplete contracting literature according to which both parties observe the default payoffs deserves more scrutiny. Similarly, Malcomson (1997) has argued that an employer may not know an employee's outside option and he remarks that little is known about hold-up under such circumstances. That asymmetric information plays a role for welfare in a hold-up model is also recognized by Gul (2001), Lau (2008), and Sloof (2008), who also provides experimental evidence.

[4] For instance, the seller may be privately informed about the probability of finding an alternative trading partner, or about the difficulty to adapt the intermediate good to another buyer's needs, or about his ability to use the intermediate good himself to produce a final good. See also Schmitz (2006) for a related model in which the seller learns the fraction of the surplus the he can realize on his own *after* the investment is sunk, so that no signaling can occur.

this effect indeed can mitigate the hold-up problem. We find that the outside option signaling game has an essentially unique equilibrium outcome. All perfect Bayesian equilibria of the game with an arbitrarily fine grid of possible types lead to the same payoffs and distribution of investments.

If the seller's maximum possible outside option is known to be relatively low compared to the value of the investment within the relationship, all types of sellers invest the same amount. Specifically, they choose the investment level that the type with the maximum outside option would choose under symmetric information. Clearly, in such a pooling equilibrium ex post efficiency is achieved and investments and joint surplus are higher than in the case with complete information.

In general, however, the equilibrium is a hybrid (semi-pooling) equilibrium. There is a cut-off type such that all sellers with a lower outside option pool on this type's strategy. This cut-off type, and all higher ones, mix between their own and all higher types' complete information investments.[5] While the information asymmetry leads to higher investments, this comes at the expense of the ex post inefficiencies which occur when the buyer, who mixes between different offers, mistakenly tries to call the seller's bluff by making an offer that is smaller than the seller's outside option. How the joint surplus compares to the case with complete information therefore depends on the parameters of the model.

The outside option signaling game that we introduce in this paper is quite distinct from standard signaling games, because the cost of the signal is constant across types, and the benefit depends only indirectly on types. Specifically, types only matter if the uninformed buyer makes a sufficiently low offer, so that ex post inefficient separation occurs.[6] Moreover, different types of sellers would choose

[5]A characteristic of our signaling model is hence a "bluffing" element that leads to an equilibrium in mixed strategies. The fact that the equilibrium is in mixed strategies due to a commitment problem is somewhat reminiscent of equilibria in hold-up problems with unobservable investments as studied in Gul (2001) and Gonzales (2004). Yet, note that in contrast to these papers we follow the incomplete contracting literature in assuming that investments are observable.

[6]Spence (2002) contains an example with similar features, in which firms can learn a worker's productivity at a cost. In this case, high productivity types separate by moving to firms that learn the type, and low types pool in a firm that does not learn. Other papers that consider productive signaling include Hermalin (1998), in which a leader may signal a worthwhile project by exerting high effort, and Daughety and Reinganum (2009), in which a signaling motive helps a team to overcome a free-riding problem.

different levels of investment if information was symmetric. Finally, while signaling games are typically plagued by a multiplicity of equilibria, refinements to pin down beliefs following zero probability events are not needed in our model.

While for the main part of the paper we follow the incomplete contracting literature in assuming that no contracts are written before the investment is made, we also explore the consequences of ex ante contracting when investments are verifiable. In this case, the buyer can offer a menu of contracts that require a certain level of investment. The main result is that the investment is always set at the first-best level and the optimal contract uses different separation probabilities as the unique screening device. Whenever the optimal contract specifies a positive probability of separation for some types, then these types *overinvest* given how the investment is later used, because conditional on taking the outside option with a positive probability, efficiency implies an investment level lower than the first-best one. The result thus adds a new twist to the literature on screening models with type-dependent outside options (Moore 1985, Lewis and Sappington 1989, Maggi and Rodriguez-Clare 1995, Jullien 2000, Rasul and Sonderegger 2010).

The remainder of the paper is organized as follows. In Section II, the outside option signaling game is introduced. In Section III, we first go through the special case of two possible types in order to illustrate the kind of equilibria that we find also in the general cases of a finite type space and a continuum of types. While it is very natural to think about the problem using a model with a finite type space, the analysis is quite technical and therefore postponed to Section VI. The results are used to find the limit equilibrium in the case of an atomless distribution, which is introduced in Section IV. The screening version of the model, in which the buyer can offer a menu of contracts with contractible investment levels, is analyzed in Section V. Proofs are relegated to an appendix.

II The model

The model describes an interaction between a buyer and a seller.[7] We first describe and solve the game with complete information and then introduce asymmetric information.

In the game with *complete information*, the seller chooses an investment $i \in \mathrm{I}$, at cost $c(i)$, to improve the value of an intermediate good or a service to be traded. If seller and buyer agree on trade, they can together generate a value of $v(i)$, while the value that the seller can realize without the buyer is only the fraction $\theta v(i)$, where $\theta \in \Theta \subset [0,1]$.[8] The buyer observes the investment and thus the value of the good and makes an offer about how to share the surplus with the seller. If the seller rejects the offer, he gets $\theta v(i)$ from taking his outside option, while the buyer makes zero profit. If the seller accepts, they split the generated surplus as proposed by the buyer.

Throughout, we make the following assumptions:

Assumption 1. *Let* $\mathrm{I} = \mathbb{R}_+$, *and let the functions* v *and* c *be differentiable, increasing, and concave resp. strictly convex. Furthermore* $v(0) \geq 0$, $c(0) = 0$, $c'(0) = 0$, *and* $\lim_{i\to\infty} c'(i) = \infty$.

It is assumed that the parties cannot write a contract ex ante. After having observed the chosen investment level, the buyer can make a take-it-or-leave-it offer to the seller. If θ is the type of the buyer, i the seller's investment, $o \in [0,1]$ the buyer's offer, expressed as a share of the surplus, and $a \in \{0,1\}$ the acceptance decision of the seller, then the seller's payoff is given by

$$\big(ao + (1-a)\theta\big)v(i) - c(i) \tag{1}$$

and the buyer's payoff by

$$a(1-o)v(i). \tag{2}$$

The complete information game can easily be solved by backward induction. The seller will accept all offers $o > \theta$, and since the buyer could always increase her offer by an arbitrarily small amount, we

[7]The model is sufficiently abstract to also fit other settings such as an employer-employee relationship.

[8]There does not need to be a deterministic relationship between the investment and the resulting value. As long as the principal can observe the investment and the value, with some notational changes the analysis would extend to the case that $v(i)$ represents the expected value generated by investment i.

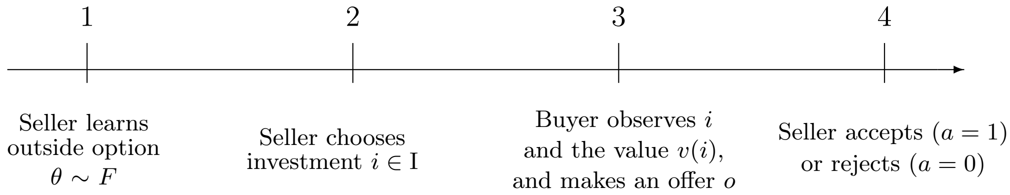

Figure 1: Timeline of the outside option signaling game.

assume that the seller accepts all offers $o \geq \theta$.[9] The buyer will offer a share θ of the realized surplus, which the seller will accept, leaving him a profit of $\theta v(i) - c(i)$ from investment i. In anticipation of this return to his investment the seller invests

$$i^c(\theta) = \arg\max_i \theta v(i) - c(i), \tag{3}$$

which given Assumption 1 always exists and is unique. Moreover, i^c is a strictly increasing function, which implies that its inverse exists, which we denote by $\theta^c : i^c(\Theta) \to \Theta$. Hence, there is underinvestment compared to the first best investment level $i^c(1)$, which maximizes the net surplus

$$S(i) = v(i) - c(i). \tag{4}$$

The seller's payoff under complete information, in dependence on the outside option θ, is denoted by

$$u^c(\theta) = \max_i \theta v(i) - c(i). \tag{5}$$

Note that u^c is increasing and strictly convex.[10] If the type space Θ is an interval, then the derivative of u^c is equal to $v \circ i^c$.

Next, consider the game with *incomplete information*, where θ is private information of the seller. The sequence of events is illustrated in Figure 1. We assume that first the seller learns his type θ,

[9]This holds for all types except $\theta = 1$. Since the buyer makes no profit on this type, it does not matter whether we assume that this type rejects or accepts an offer of 1.

[10]We could alternatively make this assumption directly or replace the conditions in Assumption 1 by other conditions from which it follows. That is, investment decisions can be allowed to be multi-dimensional or discrete as long as the optimal investment levels lead to an increasing and strictly convex function u^c.

which is drawn from the type space $\Theta \subset [0,1]$ according to a distribution function F. Throughout the paper, we make the following assumption.

Assumption 2. *F is log-concave.*[11]

The buyer only knows the distribution of the outside option, but not the realized value. She observes the seller's investment, forms beliefs about the outside option and then makes a take-it-or-leave-it offer that is optimal for her given her updated beliefs about the acceptance threshold of the seller. We are interested in perfect Bayesian equilibria of this game. In any such equilibrium a seller of type θ will accept an offer if and only if it is greater than the outside option. We therefore fix this acceptance decision (which is the same as in the game with complete information), as the outcome following the buyer's offer. In the remainder of the paper, we then deal with the following reduced-form payoff functions: If the seller is of type θ and invests i, and the buyer makes an offer o, then the seller gets $\max(\theta, o)v(i) - c(i)$ and the buyer gets $(1-o)v(i)$ if $\theta \leq o$, and 0 otherwise.

A strategy of the seller specifies an investment for each type, possibly using a randomization device to mix over a set of investments. A strategy of the seller thus is a function $Q : \Theta \times \mathrm{I} \to [0,1]$ such that $Q(\theta, .)$, or $Q(.|\theta)$, is the distribution of investments that a type θ chooses. A strategy of the buyer maps investments into a share of the surplus that she offers to the seller, where she as well may randomize over a set of offers. While a pure strategy is given by a function from the set of investments I to the set of offers $[0,1]$, we write a mixed strategy as a function $P : \mathrm{I} \times [0,1] \to [0,1]$, where $P(i, o)$, or $P_i(o)$, is the probability that the buyer's offer, when observing investment i, is less than or equal to o.

If the buyer's strategy is given by P, a seller of type θ who chooses investment i gets the expected profit

$$U(P, i, \theta) = v(i) \int \max(\theta, o) dP_i(o) - c(i). \tag{6}$$

Given a strategy Q of the seller, the buyer's expected payoff from the pure strategy $o : \mathrm{I} \to [0,1]$ is

$$V(Q, o) = \int \int_{[\theta \leq o(i)]} (1 - o(i)) v(i) dQ(i|\theta) dF(\theta). \tag{7}$$

[11]This assumption means here that $F^{\lambda}(\theta)F^{1-\lambda}(\theta') \leq F(\lambda\theta + (1-\lambda)\theta')$ for all $\theta, \theta' \in \Theta$ and $\lambda \in [0,1]$ with $\lambda\theta + (1-\lambda)\theta' \in \Theta$.

III The two-type case

In this section, we illustrate the effects that are at work in the model by first looking at the case in which there are only two possible types, $0 < \theta_L < \theta_H < 1$. Let f_L denote the probability that the outside option is low, and $f_H = 1 - f_L$ the probability that it is high. The analysis of a more general model with more than two types (in Section VI) involves some technicalities that are absent in this special case, which nevertheless conveys much of the intuition.

We start with the buyer's offer decision. It is clear that offering any share greater than θ_H ensures acceptance, and among those offers θ_H is the most profitable one for the buyer. Similarly, any offer strictly lower than θ_L is sure to be rejected, and is thus weakly dominated by offering θ_L. Offers between θ_L and θ_H are accepted by the low type only, and θ_L is the cheapest one with this outcome. Therefore, the buyer essentially chooses between offers θ_L and θ_H according to her beliefs. Specifically, she will offer θ_H if she believes that the probability of a low outside option is smaller than $\frac{1-\theta_H}{1-\theta_L}$.

Next, consider a high-type seller. This seller type knows that for any investment i he will get $\theta_H v(i)$ ex post, given that it is never optimal for the buyer to offer more than θ_H. Therefore, he invests $i_H = \arg\max \theta_H v(i) - c(i)$. His payoff is his complete information payoff $u^c(\theta_H)$, which reflects that there is no incentive to mimic lower types in this game. Given this strategy of the high type in any possible equilibrium, it is clear that a seller with a low outside option would reveal his type if he invests any amount different from i_H. A separating equilibrium, in which the low type invests $i_L = \arg\max \theta_L v(i) - c(i)$ and is offered θ_L, cannot exist, since the low type would then have an incentive to mimic the high type and get the payoff $u^c(\theta_H)$, which is larger than $u^c(\theta_L)$. The best the low type can hope for is to pool with the high type and get $u^c(\theta_H)$. Pooling on i_H is indeed an equilibrium if the buyer makes the high offer in case both types invest i_H, i.e., if $f_L \leq \frac{1-\theta_H}{1-\theta_L}$.

If the pooling equilibrium does not exist, the only possibility left is a hybrid, or semi-pooling, equilibrium, in which the low type mixes between high and low investment. The low type is indifferent between high and low investment if the probability of offer θ_L following investment i_H is such that the low type's payoff from choosing i_H is equal to $u^c(\theta_L)$. The probability that has this property is

$$p_{HL} = \frac{u^c(\theta_H) - u^c(\theta_L)}{(\theta_H - \theta_L)v(i_H)}. \tag{8}$$

Following a low investment, the buyer offers θ_L. To make the buyer indifferent between the high and the low offer following investment i_H, the low type seller has to choose high investment with probability

$$q_{LH} = \frac{f_H(1-\theta_H)}{(\theta_H - \theta_L)f_L}. \tag{9}$$

This value is smaller than one if and only if the pooling equilibrium does not exist. This insight, that depending on the distribution there is either a pooling equilibrium or an equilibrium with mixed strategies and partial pooling, remains valid in the general case.

Observation 1. *In the two-type model, the pooling equilibrium becomes more likely the larger the probability of the high type is, and the closer together the two types are. Moreover, increasing the high type's value, or even increasing the high and the low value by an equal amount, can turn a pooling equilibrium into a semi-pooling equilibrium and thereby decrease the ex ante expected payoff of the seller.*

It is straightforward to embed the outside option signaling game into a full-fledged property rights model, where the parties are symmetrically informed before date 1, when they can agree on a simple ownership structure only. Giving the seller more property rights may then mean that θ_H and θ_L are increased. Hence, Observation 1 implies that giving the seller more property rights can be detrimental to his investment incentives, his expected payoff, and the expected total surplus, which is in stark contrast to the standard property rights model under complete information.

IV Continuum of types

In this section, we let the type space Θ be an interval, $\Theta = [\theta_L, \theta_H]$. The seller's type is drawn from the distribution F with density $f > 0$, for which the derivative f' exists.

As in the case with only two types, a fully revealing equilibrium does not exist. The reason is that in such an equilibrium, a type θ would be offered the share θ and accept. This type would invest $i^c(\theta)$

and get the payoff $u^c(\theta)$ without taking his outside option. Since any other type that deviates to $i^c(\theta)$ would get the same payoff, and u^c is increasing, lower types would have an incentive to deviate. A separating equilibrium hence does not exist, but what about a pooling equilibrium? Again as in the two-type case, the buyer will never offer a share greater than θ_H. If the seller has the highest possible outside option θ_H, he chooses $i^c(\theta_H)$ with probability one. In a pooling equilibrium all other types would have to do the same. However, types close to θ_H will invest $i^c(\theta_H)$ only if the buyer offers the share θ_H of the surplus. Whether a pooling equilibrium exists thus depends on the buyer's expected revenue from offering θ_H compared to making any other offer θ. With the definition

$$R(\theta) = (1-\theta)F(\theta), \tag{10}$$

the expected revenue from offering θ is $R(\theta)v(i^c(\theta_H))$, so that there is a pooling equilibrium if and only if $R(\theta_H) = \max_\theta R(\theta)$. This already hints at the fact that the maximizer of the function R plays an important role for the characterization of the equilibrium. Before we state the main result, we prove that this maximizer is uniquely defined.

Lemma 1. *The function R has a unique maximizer, which is denoted by $\bar{\theta}$, i.e,*

$$\bar{\theta} = \arg\max_{\theta \in [\theta_L, \theta_H]} R(\theta). \tag{11}$$

Moreover, R is weakly increasing on $[\theta_L, \bar{\theta}]$, and decreasing and strictly concave on $[\bar{\theta}, \theta_H]$.

The function R captures the tradeoff that the buyer faces, which is the tradeoff between a higher acceptance probability and a larger share of the surplus in case of acceptance. That R is increasing up to $\bar{\theta}$ implies that if in an equilibrium all types $\theta \leq \bar{\theta}$ choose the same strategy, they will be offered a share of at least $\bar{\theta}$ (which they accept). This is the case in the equilibrium of the outside option signaling game that we state in the following proposition.

Proposition 1. *An equilibrium of the outside option signaling game is given by*

$$P_i(\theta) = \begin{cases} 0 & \theta < \bar{\theta} \\ \frac{v(i^c(\theta))}{v(i)} & \bar{\theta} \leq \theta < \theta^c(i) \\ 1 & \theta^c(i) \leq \theta \end{cases} \tag{12}$$

and

$$Q(i|\theta) = \begin{cases} 0 & i < i^c(\theta) \\ 1 - \frac{(1-\theta^c(i))^2 f(\theta^c(i))}{(1-\theta)^2 f(\theta)} & i^c(\theta) \leq i < i^c(\theta_H) \\ 1 & i^c(\theta_H) \leq i \end{cases} \tag{13}$$

for all $\theta \geq \bar{\theta}$*, and* $Q(i|\theta) = Q(i|\bar{\theta})$ *for all* $\theta < \bar{\theta}$*.*

We see that as in the two type case, the seller tries to mimic higher types, never lower ones. The highest seller type chooses his complete information investment with probability 1. Any type θ between $\bar{\theta}$ and θ_H mixes between all investments in the interval $[i^c(\theta), i^c(\theta_H)]$, and chooses $i^c(\theta_H)$ with positive probability if $\theta_H < 1$. A type $\theta < \bar{\theta}$ invests in the same way as the type $\bar{\theta}$, so that investments lower than $i^c(\bar{\theta})$ never occur in equilibrium. Following this lowest investment $i^c(\bar{\theta})$, the buyer makes the offer $\bar{\theta}$ that is always accepted. For larger investments i, the buyer mixes between offers in the interval $[\bar{\theta}, \theta^c(i)]$ with an atom at $\bar{\theta}$, and thereby sometimes makes an offer that the seller does not accept.

While this result does not say that the described equilibrium is the unique outcome of the game, we show uniqueness for a finite type space in Section VI. More precisely, we show there that with a finite type space, all equilibria must lead to the same payoffs and distribution of investment. If the finite type space is understood as a partition of the interval $[\theta_L, \theta_H]$ and all functions of the finite type space are interpreted as step functions on this interval, then the functions defined in Proposition 1 are limits of sequences of such equilibrium step functions as the partition becomes arbitrarily fine.

With the explicit solution of the signaling game described in Proposition 1, we can write down the parties' payoffs and compare them to the complete information case, in which the outside option is common knowledge from the start. This case was solved as a preliminary in Section II. First, note that in the outside option signaling equilibrium, each type of seller chooses a weakly higher investment level than under complete information. The unconditional cumulative distribution function of investments is equal to $\max(0, -R' \circ \theta^c)$ for $i < i^c(\theta_H)$, and equal to 1 at $i = i^c(\theta_H)$. Since $-R'(\theta) = F(\theta) - (1-\theta)f(\theta)$, this function first order stochastically dominates the distribution of investments under complete information. However, unless the equilibrium is a pooling equilibrium ($\bar{\theta} = \theta_H$), there is also a positive probability of inefficient separation in the signaling equilibrium, and therefore we cannot conclude that the asymmetry of information in the signaling game in general leads to a higher joint surplus. Similarly, it is not possible to say anything general about the buyer's surplus in the outside option signaling game, which is equal to

$$V^* = \int_{\bar{\theta}}^{\theta_H} -R''(\theta)(1-\theta)v(i^c(\theta))d\theta + (1-\theta_H)^2 f(\theta_H)v(i^c(\theta_H)), \tag{14}$$

compared to the buyer's surplus under complete information, which is equal to $V^c = E[(1-\theta)v(i^c(\theta))]$.

We can, however, say something about the seller's payoff. In the outside option signaling game, a seller with outside option θ gets $u^c(\max(\theta, \bar{\theta}))$, i.e., the seller's ex ante expected profit is

$$U^* = F(\bar{\theta})u^c(\bar{\theta}) + \int_{\bar{\theta}}^{\theta_H} u^c(\theta)dF(\theta). \tag{15}$$

This is larger than the seller's expected payoff under complete information, which is $U^c = E[u^c(\theta)]$.

To illustrate that the buyer's payoff and the joint payoff in the signaling equilibrium can be larger or smaller than the corresponding payoffs under complete information, Table 1 shows these values in four examples that differ with respect to the distribution of types. With a uniform distribution, the buyer's payoff happens to be equalized in the two regimes. A pdf $f(\theta) = 6\theta(1-\theta)$ puts more weight on intermediate types, which is beneficial for the buyer, who has to give up a large share of the surplus to a high type seller and suffers from the low investment of low type sellers. This advantage bears out to a larger extent under complete information than under signaling. A pdf $f(\theta) = 3(1-\theta)/4\sqrt{\theta}$ puts more weight on lower types. Since lower types invest very little under complete information, here the signaling equilibrium, in which low types are encouraged to invest, implies a larger payoff for the buyer.

We can also compare the seller's payoff in the signaling game to other scenarios regarding the distribution of information and timing. Consider first a scenario in which the outside option becomes common knowledge after the investment is sunk, and is not known before to any party. In this case, there are no ex post information rents since the buyer's offer equals the true value of the outside option, and at the same time the seller cannot tailor his investment decision to the outside option. Instead, he maximizes his expected payoff $E[\theta]v(i) - c(i)$ over i. With the resulting payoff $u^c(E[\theta])$, the seller is worse off than he would be even in the complete information case.

$f(\theta)$	V^*	$V^* + U^*$	V^c	$V^c + U^c$
$\frac{3(1-\theta)}{4\sqrt{\theta}}$	0.22	0.29	0.11	0.16
1	0.17	0.38	0.17	0.33
$6\theta(1-\theta)$	0.18	0.39	0.2	0.35
$30\theta^4(1-\theta)$	0.11	0.44	0.18	0.45

Table 1: This table shows the (rounded) buyer's payoff and the joint surplus in the signaling equilibrium (V^* and $V^* + U^*$) as well as the same quantities for complete information (V^c and $V^c + U^c$) for the specification $v(i) = i$, $c(i) = \frac{1}{2}i^2$, $\Theta = [0, 1]$ and $f(\theta)$ as indicated in the first column. To apply our results for the different distributions, we can either show that Assumption 2 (log-concavity) holds, or show directly that the conditions in Lemma 1 hold.

We can also compare our results to a timing as in Schmitz (2006), in which the seller (and only the seller) learns the outside option once the investment is sunk. In this case, there is no signaling motive, and the seller's choice of investment is independent of his type. Consequently, the buyer makes an offer of $\bar{\theta}$ and the seller invests $i^c(E[\max(\theta, \bar{\theta})])$. While the investment is higher than in the scenario above, it is not always put to its best use, as all types above $\bar{\theta}$ reject the offer. The seller gets $u^c(E[\max(\bar{\theta}, \theta)])$ which is more than in the previous case, as he enjoys some information rents. Nevertheless, the seller is still better off in the signaling equilibrium, which allows him to both tailor the investment to the outside option and earn some information rents. Since of all the possible scenarios, the seller's payoff is highest in the signaling equilibrium, he would influence the timing or information distribution in the direction of the signaling structure whenever possible. This is summarized in the following observation.

Observation 2. The seller has an incentive to learn the outside option early and let it be known that he knows about his outside option.

Finally, we can now revisit the question of the effect of giving the seller more property rights, which we think of as a first order stochastic dominance shift in the distribution of outside options. First, we consider only the change in the cut-off value that results from a change in the distribution function. If $\bar{\theta}$ increases, then all types with an outside option smaller than the cut-off value, who get $u^c(\bar{\theta})$, are strictly better off. Types larger than the new cut-off value get the same payoff $u^c(\theta)$ as before.

Observation 3. If the cut-off value $\bar{\theta}$ increases, each seller type is weakly better off. If a first order stochastic dominance shift of the distribution of outside options increases the cut-off value, then it also increases the seller's ex ante payoff.

This observation tells us that if sellers come from two different populations with distribution of types F and $\tilde{F}$, respectively, where $\tilde{F}$ first order stochastically dominates F, then if the cut-off value is higher under $\tilde{F}$ than under F, the seller's ex ante payoff must be higher under $\tilde{F}$ than under F. This means that if there is some observable characteristic that implies a higher outside option on average, then low types can benefit from belonging to this group as they can hide behind the better average bargaining position in their group and get a good offer.

Recall how in the case with only two types in Section III a decrease in $\bar{\theta}$, which meant a change from a pooling to a semi-pooling equilibrium, could easily happen with first order stochastic dominance shifts in the distribution. While this same effect can still be constructed here, by removing some mass at types slightly lower than $\bar{\theta}$ and adding it to types slightly larger than $\bar{\theta}$, it now seems more likely that more property rights would increase $\bar{\theta}$ and make the seller better off. For example, if F is a uniform distribution on an interval $[a, b]$, then $\bar{\theta} = \min(b, \frac{a+1}{2})$. If we increase a or b, then $\bar{\theta}$ also increases.

V Contractible investments

In the game that is studied in the main part of this paper, all the buyer can do is to make a take-it-or-leave-it offer based on her updated beliefs. This is optimal for her from an ex post perspective, but not necessarily from an ex ante perspective. In this section, we explore the consequences of full commitment and ask what would happen if the buyer could offer a binding contract conditional on investment before the seller moves. We maintain the assumption that the seller's type is not observable, and characterize the optimal screening contract.

We use the revelation principle and let a general contract be a map from types into outcomes that satisfies the incentive compatibility constraints of each type of seller telling the truth. In addition, the buyer has to take into account that the seller can take his outside option. All that matters for truth telling and participation of the seller is his expected payoff following his announcement. Therefore, it is sufficient to focus on contracts of the form $(t(\theta), i(\theta), x(\theta))$, where $t(\theta)$ is an unconditional payment from the seller to the buyer that an announced type θ is required to make, $i(\theta)$ is the required investment, and $x(\theta)$ the probability of separation. We first allow for two different points in time when separation can occur and let $x(\theta) = (x_1(\theta), x_2(\theta))$, where $x_1(\theta)$ is the probability of separation before the investment is made, and $x_2(\theta)$ is the probability of separation after the investment is made. Hence, first the seller makes the payment $t(\theta)$, then with probability $x_1(\theta)$, the relationship ends directly after the seller has made his payment, leaving the seller with payoff $u^c(\theta) - t(\theta)$. While we allow the possibility of such an early break-up of the relationship, in an optimal contract it will be true that $x_1(\theta) = 0$. With probability $1 - x_1(\theta)$, the seller makes the investment $i(\theta)$, and then with probability $1 - x_2(\theta)$, buyer and seller collaborate and the seller gets the whole ex post surplus $v(i(\theta))$. There is no loss of generality in assuming this form of contracts, since all payoff transfers from the seller to the buyer can be handled by the payment $t(\theta)$. Given such a contract, the expected payoff of a seller of type θ who pretends to be of type $\tilde{\theta}$ is

$$(1 - x_1(\tilde{\theta}))\Big(S(i(\tilde{\theta})) - x_2(\tilde{\theta})(1-\theta)v(i(\tilde{\theta}))\Big) + x_1(\tilde{\theta})u^c(\theta) - t(\tilde{\theta}), \tag{16}$$

where the definition $S(i) = v(i) - c(i)$ was used. A truth-telling seller gets the payoff

$$u_S(\theta) = (1 - x_1(\theta))\Big(S(i(\theta)) - x_2(\theta)(1-\theta)v(i(\theta))\Big) + x_1(\theta)u^c(\theta) - t(\theta). \tag{17}$$

The buyer's optimization problem is the following:

$$\max \int_{\theta_L}^{\theta_H} t(y)dF(y) \tag{18}$$

subject to the incentive compatibility constraint

$$u_S(\theta) \geq u_S(\tilde{\theta}) + (1 - x_1(\tilde{\theta}))(\theta - \tilde{\theta})x_2(\tilde{\theta})v(i(\tilde{\theta})) + x_1(\tilde{\theta})(u^c(\theta) - u^c(\tilde{\theta})) \tag{IC}$$

and the individual rationality constraint

$$u_S(\theta) \geq u^c(\theta), \tag{IR}$$

which have to hold for all $\theta, \tilde{\theta} \in [\theta_L, \theta_H]$.

We will show next that an optimal contract will specify the first best investment level $i^c(1)$ and $x_1(\theta) = 0$. To see this, consider any contract $(t(\theta), i(\theta), x(\theta))$. We then define the contract $(\tilde{t}(\theta), \tilde{i}(\theta), \tilde{x}(\theta))$ as

$$\tilde{t}(\theta) = t(\theta) + S(i^c(1)) - (1 - x_1(\theta))S(i(\theta)) - x_1(\theta)S(i^c(\theta)), \quad (19)$$

$$\tilde{i}(\theta) = i^c(1), \quad (20)$$

$$\tilde{x}_1(\theta) = 0, \text{ and} \quad (21)$$

$$\tilde{x}_2(\theta) = x_1(\theta)\frac{v(i^c(\theta))}{v(i^c(1))} + (1 - x_1(\theta))x_2(\theta)\frac{v(i(\theta))}{v(i^c(1))} \in [0, 1]. \quad (22)$$

With this new contract, a truth telling seller's payoff is $S(i^c(1)) - \tilde{x}_2(\theta)(1 - \theta)v(i^c(1)) - \tilde{t}(\theta)$, which is equal to $u_S(\theta)$ under the old contract. Hence, the individual rationality constraint (IR) is satisfied also for the new contract. The incentive constraint (IC) now reads

$$\begin{aligned} u_S(\theta) &\geq u_S(\tilde{\theta}) + (\theta - \tilde{\theta})\tilde{x}_2(\tilde{\theta})v(i^c(1)) \quad (23) \\ &= u_S(\tilde{\theta}) + (1 - x_1(\tilde{\theta}))(\theta - \tilde{\theta})x_2(\tilde{\theta})v(i(\tilde{\theta})) + x_1(\tilde{\theta})(\theta - \tilde{\theta})v(i^c(\tilde{\theta})). \end{aligned}$$

Because $u^c(\theta)$ is a convex function with derivative $v(i^c(\theta))$, it follows from the old contract's incentive constraint that this constraint is satisfied as well. Moreover, this new contract generates higher expected profit for the buyer because $\tilde{t}(\theta) \geq t(\theta)$. Thus, we have shown that $i(\theta) = i^c(1)$ and $x_1(\theta) = 0$. In order to find the buyer's optimal separation probabilities $x_2(\theta)$ and corresponding payments $t(\theta)$, where a higher separation probability corresponds to a lower up-front payment, we use standard tools from mechanism design and the literature on type-dependent outside option.[12] In this literature, the difficulty mostly lies in finding out for which types the individual rationality constraint binds. Here, the proof of the following proposition shows that the IR constraint is binding for the whole interval of types that are larger than the cut-off value $\bar{\theta}$.

[12]See e.g. Jullien (2000) for a very general treatment, which nevertheless does not encompass our model as a special case.

Proposition 2. *An optimal screening contract specifies the investment level* $i^c(1)$*. Sellers of type* $\theta \leq \bar{\theta}$ *choose to collaborate ex post with probability* 1 *and pay* $t(\theta) = S(i^c(1)) - u^c(\bar{\theta})$ *up-front, which leaves them with a payoff of* $u^c(\bar{\theta})$*. Types* $\theta > \bar{\theta}$ *separate with probability* $x_2(\theta) = \frac{v(i^c(\theta))}{v(i^c(1))}$ *and pay up-front* $t(\theta) = S(i^c(1)) - S(i^c(\theta))$*. These seller types get the payoff* $u^c(\theta)$*.*

While every seller type receives the same payoff as in the signaling equilibrium, the buyer's expected payoff and the joint surplus are obviously higher than in the case without commitment. Since now i is verifiable and a hold-up problem does not exist, it may seem intuitive that an optimal contract specifies the investment $i^c(1)$ for all types: Since seller types differ only with respect to the outside option, the screening device is the probability of separation, not the investment. But if the asset is used outside the relationship with positive probability, then the value $i^c(1)$ is not the optimal investment. Instead, the optimal investment for type θ is $i^c(1 - x_2(\theta) + x_2(\theta)\theta)$.

Observation 4. *Any type* θ *in the open interval* $(\bar{\theta}, 1)$ *overinvests: given that the separation probability* $x_2(\theta)$ *is positive for these types, the efficient investment* $i^c(1 - (1 - \theta)x_2(\theta))$ *is strictly smaller than* $i^c(1)$*.*

Note that an overinvestment effect is also present in the signaling model, where it counteracts the hold-up effect on investment. With contractible investments, the hold-up problem is absent and only the overinvestment effect is present.

VI Finite type space

In this section, we analyze the outside option signaling game with $\Theta = \{\theta_1, ..., \theta_H\}$ where $0 \leq \theta_1 < \theta_2 < ... < \theta_H < 1$.[13] We use the shortcut $i_k = i^c(\theta_k)$. Let (P, Q) be a perfect Bayesian equilibrium of the outside option signaling game. In the following, we will derive properties of (P, Q), in order to eventually arrive at a characterization of all equilibrium outcomes. Let I* be the set of investments that

[13]The assumption $\theta_H < 1$ is made for simplicity. We could easily add types $\theta \geq 1$ who would always invest $i^c(\theta)$ and get no acceptable offer from the buyer. That is, a type $\theta \geq 1$ seller would neither mimic other types nor be mimicked himself.

are chosen with positive probability in the equilibrium (P, Q), and let $\Theta^*(i)$ denote the set of all types that choose $i \in \mathrm{I}^*$ with positive probability. We denote by $u^*(\theta)$ the equilibrium payoff received by a seller of type θ, so that with this notation we have for all $i \in \mathrm{I}^*$ and $\theta \in \Theta^*(i)$ that $u^*(\theta) = U(P, i, \theta)$.

Note first that $u^*(\theta) \geq u^c(\theta)$, because a type θ can always guarantee himself the payoff $u^c(\theta)$ independent of the buyer's behavior, by investing $i^c(\theta)$ and taking his outside option. Similarly, because the seller's payoff is weakly increasing in θ for all offers and investments, $U(P, i, \theta)$ and $u^*(\theta)$ are weakly increasing in θ. A higher type could always play a lower type's strategy and get at least the same payoff as that type.

In the following, we will first show (Lemma 2) that if an investment i occurs at all in equilibrium, then it is chosen with positive probability by the type $\theta^c(i)$ that chooses i under symmetric information, and by none of the higher types. Then, in Lemma 3, we show that investing i is optimal for all lower types, i.e. those between θ_1 and $\theta^c(i)$. Finally, in Proposition 3 we will answer the question which investments will be chosen in equilibrium. The reader who is not interested in the proofs may skip the lemmas leading to Proposition 3, which contains the main result of this section.

Lemma 2. *For all $i \in \mathrm{I}^*$ it holds that $\theta^c(i) = \max \Theta^*(i)$.*

In particular, only investments in the set $\{i_1, ..., i_H\}$ are chosen at all. We can use the one-to-one relationship between θ_k and i_k and express everything in types. This highlights that in this model types are distinguishable by their investment in the complete information case. We can also identify the buyer's offer with the type that just accepts it, and then write the equilibrium strategies as matrices P and Q. An entry p_{kl} in the matrix P stands for the probability of offer θ_l when investment i_k is observed, and an entry q_{kl} in Q is the probability of type k investing i_l, or "mimicking" type l. Since we have shown that in any equilibrium the mixed strategy of type θ_k has support in $\{i_k, ..., i_H\}$ and the buyer's random offer following investment i_k takes on values in $\{\theta_1, ..., \theta_k\}$, equilibrium strategies P and Q are triangular matrices. Equilibrium conditions for strategies (P, Q) in matrix form then look as follows:

$u^c(\theta_k)$	$\max_i v(i)\theta_k - c(i)$
$i_k = i^c(\theta_k)$	$\arg\max_i v(i)\theta_k - c(i)$
θ^c	inverse of i^c
q_{kl}	probability that type θ_k chooses investment i_l
p_{lk}	probability that offer is θ_k when investment is i_l
$P_{i_l}(\theta_k)$	probability that offer is $\leq \theta_k$ when investment is i_l
$Q(i_l\|\theta_k)$	probability that type θ_k's investment is $\leq i_l$
$u^*(\theta_k)$	type θ_k's payoff in the equilibrium (P, Q)
$\Theta^*(i_l)$	set of all types θ_k with $q_{kl} > 0$
I^*	set of all investments i_l with $q_{kl} > 0$ for some k

Table 2: Some notation for the finite type case.

- $q_{kl} > 0$ implies that

$$l \in \arg\max_m v(i_m) \sum_{j=1}^{m} p_{mj} \max(\theta_j, \theta_k) - c(i_m), \tag{24}$$

- for each l with $i_l \in \mathrm{I}^*$, $p_{lj} > 0$ implies that

$$j \in \arg\max_m (1 - \theta_m) \sum_{k=1}^{m} f_k q_{kl}. \tag{25}$$

This notation is summarized in Table 2. We will show next that a given type θ_k's set of best responses to the buyer's strategy P includes all investments that are greater than or equal to i_k and are chosen at all in the equilibrium. In other words, if an investment i_k is chosen at all, then it is the optimal choice for every type not greater than the corresponding type θ_k.

Lemma 3. *For all $i_k \in \mathrm{I}^*$ it holds that $U(P, i_k, \theta) = u^*(\theta)$ for all $\theta = \theta_1, ..., \theta_k$.*

We have shown so far that, while there may be investments that do not occur at all in equilibrium, every investment that does occur is chosen by the type that would invest the same amount with symmetric information. Furthermore, all lower types' payoff from choosing this investment equals their equilibrium payoff. In order to be consistent with this structure, the buyer's strategy must induce all these indifferences. This observation gives rise to the following lemma.

Lemma 4. *For all k and $i_m \in \mathrm{I}^*$ with $m > k$ it holds that*

$$P_{i_m}(\theta_k)v(i_m) = \frac{u^*(\theta_{k+1}) - u^*(\theta_k)}{\theta_{k+1} - \theta_k}. \tag{26}$$

Moreover, for all $i_m, i_k \in \mathrm{I}^*$ with $m \geq k$ it holds that $p_{mk} > 0$.

Now that we have some idea about the offers that the buyer must be willing to make, we turn to a description of the buyer's behavior, in order to pin down the seller's equilibrium strategy. As in the continuous case, the function $R(\theta) = (1-\theta)F(\theta)$ plays a role here. Similar to Lemma 1 it can be shown here that if we define

$$\bar{k} = \max\{k \in \{1, ..., H\} : \; R(\theta_k) \geq R(\theta_{k-1})\}, \tag{27}$$

the function R is weakly increasing on $\{\theta_1, ..., \theta_{\bar{k}}\}$, strictly decreasing on $\{\theta_{\bar{k}}, ..., \theta_H\}$, and the linear interpolation of the points $(\theta_{\bar{k}}, R(\theta_{\bar{k}})), ..., (\theta_H, R(\theta_H))$ is concave.

To understand the role of R, assume for a moment that all types choose the same investment. Then $R(\theta)$ describes the buyer's expected share of the surplus if she makes a take it or leave it offer of θ. The maximum $\theta_{\bar{k}}$ of this function is the offer that she would make in a pooling equilibrium. Can a pooling equilibrium exist? Since the highest type θ_H chooses i_H in any equilibrium, if all types pool on the same investment, it must be on i_H. It follows that there is such a pooling equilibrium if and only if $\theta_{\bar{k}} = \theta_H$. Moreover, this suggests that in any equilibrium, pooling is only possible for types lower than $\theta_{\bar{k}}$. Since a separating type could easily be mimicked by a lower type, equilibria must typically be in mixed strategies.

Proposition 3. *Any perfect Bayesian equilibrium of the outside option signaling game must have the following form: No investment below $i_{\bar{k}}$ is chosen. A type θ_k with $k \geq \bar{k}$ mixes between all investments in $\{i_k, ..., i_H\}$, with expected payoff equal to $u^c(\theta_k)$. All types θ_k with $k < \bar{k}$ mix over $\{i_{\bar{k}}, ..., i_H\}$ with payoff $u^c(\theta_{\bar{k}})$. When observing investment i_k, the buyer mixes between offers in $\{\theta_{\bar{k}}, ..., \theta_k\}$, and her expected payoff from any such offer is $(1-\theta_k)v(i_k)$.*

This result is a uniqueness result in the sense that in any perfect Bayesian equilibrium of the game, payoffs of the buyer and the seller are uniquely determined. Refinements to pin down beliefs following zero probability events are not needed for this result. This is unusual for a signaling game and is due to the special structure of this game, in which the buyer's offers only matter to a limited extent for the seller's payoff. Equilibrium investment in fact turns out to be a poor signal for a high outside option. The types that pool never reveal their outside options, and the others do not improve their payoff in the signaling game compared to what they could get independent of the buyer. These higher types separate in the sense that they choose different strategies. Because of the randomization, however, a chosen investment does not give away the type ex post. An observed investment could have been chosen by any type who would invest weakly less under complete information.

From all the indifference conditions that have to be met in an equilibrium we are able to obtain an equilibrium candidate. Combining Proposition 3 and Lemma 4 yields for all $k \geq \bar{k}$ and $m > k$

$$P_{i_m}(\theta_k) = \frac{u^c(\theta_{k+1}) - u^c(\theta_k)}{(\theta_{k+1} - \theta_k)v(i_m)} \quad \text{and } P_{i_k}(\theta_k) = 1, \tag{28}$$

as well as for $k < \bar{k}$

$$P_{i_m}(\theta_k) = 0. \tag{29}$$

For the definition of the seller's strategy, we assume that all types $j < \bar{k}$ pool on type $\bar{k}$'s strategy:

$$q_{jk} = q_{\bar{k}k} \qquad \text{for all } j < \bar{k}. \tag{30}$$

Let us further define $\lambda_k = \frac{f_k(1-\theta_k)(1-\theta_{k-1})}{\theta_k - \theta_{k-1}}$ and $\lambda_{H+1} = 0$ and

$$q_{\bar{k}k} = \frac{\lambda_k - \lambda_{k+1}}{R(\theta_{\bar{k}})} \qquad \text{for all } k > \bar{k} \tag{31}$$

$$q_{\bar{k}\bar{k}} = 1 - \frac{\lambda_{\bar{k}+1}}{R(\theta_{\bar{k}})} \tag{32}$$

$$q_{jk} = \frac{\lambda_k - \lambda_{k+1}}{\lambda_j} \qquad k \geq j > \bar{k} \tag{33}$$

Proposition 4. *The strategies described in equations (28), (29), (30), (31), (32) and (33) form an equilibrium of the outside option signaling game.*

VII Conclusion

In this paper, we have introduced ex ante private information about an agent's reservation value in the kind of hold-up problem that is at the center of the literature started by Grossman and Hart's (1986) seminal work on the pros and cons of vertical integration. The resulting outside option signaling game has interesting features which make it quite distinct from other signaling models. The simplicity of the model allows us to fully characterize the resulting equilibrium payoffs, which are uniquely determined. The equilibrium involves pooling up to a certain type of outside option, such that all lower types get the same payoff and because they accept all offers in equilibrium, these types are not distinguishable, even ex post. Higher types follow a mixed strategy and on average obtain the same payoff as with complete information. The fact that the seller randomizes between investment levels reflects that there is a strong force against a separating equilibrium in this model: If an investment is only chosen by high types and triggers high offers, this investment becomes attractive for lower types as well.

In the outside option signaling game, there is a gap between the chosen investment and the investment that would result if the seller would get the full return to his investment. We have shown that this gap vanishes if investment is verifiable. This gap would also shrink if the seller had greater bargaining power than in the game that was analyzed. For example, if the bargaining game was modeled as the seller making a take-it-or-leave-it offer with probability α and the buyer only with probability $1-\alpha$, then a higher α would increase the surplus and the seller's payoff. Although it is standard in principal-agent models to assume take-it-or-leave-it offers by the principal, it would be interesting to allow for more complex bargaining games at the ex post stage. While the results should be the same if the buyer was able to make repeated offers, results are likely to change and become difficult to obtain if both players made offers.

Our model of a one-shot buyer-seller interaction makes the prediction of higher rates of separation when relationship-specific investment is higher. Two kinds of relationships can arise: Stable relationships that are characterized by low investments and low profits ($\theta \leq \bar{\theta}$), and unstable relationships that are characterized by high investment and high separation rates ($\theta > \bar{\theta}$). However, there are ways in which the parties might try to mitigate the hold-up problem, say by establishing repeated interactions, and these factors could lead to a positive instead of a negative correlation between the stability of the relationship and the level of investment. It might therefore be interesting to extend the analysis to take into account dynamic considerations and/or competition between buyers.

There are a couple of other extensions of the model that are promising. One interesting task for future research is to allow the payoff that the buyer gets when the seller takes the outside option to depend on the seller's type. This might admit an even greater set of applications, for instance the interpretation of the outside option as suing for payment, with private information about the probability of winning.[14] Another possible avenue for future research is to focus on the case of pure rent-seeking, in which the investment increases the outside value but is of little use inside the relationship. Investment can then still be used as a signal for profitable outside opportunities, but higher investment is no longer more efficient.

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[14]See Choné and Linnemer (2010) for a related model in the context of pretrial bargaining and investment in trial preparation.

A Proofs

Proof of Lemma 1. To show that this property of R follows from log-concavity of F, we show first that wherever R is convex, it must be strictly increasing:

$$R''(\theta) \geq 0 \Rightarrow R'(\theta) > 0. \tag{34}$$

The first derivative of R is

$$R'(\theta) = (1-\theta)f(\theta) - F(\theta), \tag{35}$$

and the second derivative is

$$R''(\theta) = (1-\theta)f'(\theta) - 2f(\theta). \tag{36}$$

Assume now that $R''(\theta) \geq 0$. This implies that $f'(\theta) > 0$ and $(1-\theta) \geq \frac{2f(\theta)}{f'(\theta)}$. Because log-concavity means $F(\theta)f'(\theta) \leq f(\theta)^2$ we also have

$$R'(\theta) \geq \frac{2f(\theta)^2 - F(\theta)f'(\theta)}{f'(\theta)} \geq \frac{f(\theta)^2}{f'(\theta)} > 0. \tag{37}$$

Hence, we have shown that property (34) holds. This property implies that the function R can have no interior minimum (i.e., no point with $R'(\theta) = 0$ and $R''(\theta) \geq 0$). We also know that R is nonnegative with $R(\theta_L) = 0$. Therefore, the global maximum at $\bar{\theta}$ is also the unique local maximimum. The function R is weakly increasing up to the point $\bar{\theta}$ and weakly decreasing for all $\theta \geq \bar{\theta}$. Because $R'(\theta) \leq 0$ for all $\theta \geq \bar{\theta}$, it follows again from property (34) that the function R is strictly concave on that range. □

Proof of Proposition 1. In a first step, we show that the functions P_i and $Q(.|\theta)$ are indeed distribution functions. The function P_i has $P_i(\theta) = 0$ for all $\theta_L \leq \theta < \bar{\theta}$ and $P_i(\theta) = 1$ for all $\theta \geq \theta^c(i)$. It is nondecreasing inbetween because $v \circ i^c$ is increasing. Similarly, the function $Q(.|\theta)$ has $Q(i|\theta) = 0$ for all $i < i^c(\bar{\theta})$ and $Q(i^c(\theta_H)|\theta) = 1$. It is nondecreasing because θ^c is increasing in i and the derivative

$$\frac{\partial}{\partial y}\left(1 - \frac{(1-y)^2 f(y)}{(1-\theta)^2 f(\theta)}\right) = -\frac{(1-y)R''(y)}{(1-\theta)^2 f(\theta)} \tag{38}$$

is positive on the relevant range, since R is concave on the interval $[\bar{\theta}, \theta_H]$.

As a second step, we show that $P_i(\theta) = 1$ for all $\theta \geq \theta^c(i)$ is part of the buyer's best response to a seller strategy with $Q(i|\theta) = 0$ for all $i < i^c(\theta)$, and vice versa: If an investment i is never chosen by a seller of type higher than $\theta^c(i)$, the buyer optimally never offers more than $\theta^c(i)$ when observing i. Conversely, because the buyer, when observing an investment i, never offers more than $\theta^c(i)$, types $\theta > \theta^c(i)$ would get only $\theta v(i) - c(i)$ by choosing i and therefore prefer the investment $i^c(\theta)$ over i.

In the third step we show that all investments in the support of $Q(.|\theta)$ are best replies to the buyer's strategy. First we look at a seller of type $\theta \geq \bar{\theta}$. Such a seller's expected payoff from choosing an investment $i^c(\theta_H) \geq i \geq i^c(\theta)$ is

$$v(i) \int \max(\theta, y) dP_i(y) - c(i), \tag{39}$$

which is the same as

$$v(i) \left(\theta P_i(\theta) + \int_{(\theta, \theta^c(i)]} y dP_i(y) \right) - c(i). \tag{40}$$

Because P_i is continuous on the interval $[\theta, \theta^c(i)]$ we can use integration by parts to evaluate this integral as

$$v(i) \left(\theta P_i(\theta) + \theta^c(i) P_i(\theta^c(i)) - \theta P_i(\theta) - \int_{\theta}^{\theta^c(i)} P_i(y) dy \right) - c(i), \tag{41}$$

which is equal to

$$v(i)\theta^c(i) - \int_{\theta}^{\theta^c(i)} v(i^c(y)) dy - c(i). \tag{42}$$

Since $v(i)\theta^c(i) - c(i) = u^c(\theta^c(i))$ and since the derivative of u^c is $v \circ i^c$, this is the same as $u^c(\theta)$. Hence, a seller of type $\theta \geq \bar{\theta}$ in expectation gets his complete information payoff following any investment $i \in [i^c(\theta), i^c(\theta_H)]$.

Next, we consider seller types in the interval $[\theta_L, \bar{\theta}]$. Since $P_i(\theta) = 0$ for all $\theta < \bar{\theta}$, i.e., the buyer never makes an offer that is smaller than $\bar{\theta}$, all types in this interval accept all offers and therefore they all have the same expected payoff following any investment they choose. Like the type $\bar{\theta}$, seller types in this interval are indifferent between investments in $[i^c(\bar{\theta}), i^c(\theta_H)]$.

As the last step, it remains to show that all offers in the support of P_i are best responses to the mixed strategy of the seller. Using Bayes' Law, this means that we have to show that for all

$\bar{\theta} \leq \theta \leq \theta^c(i)$ it holds that

$$(1-\theta)\,\frac{\int_{\theta_L}^{\theta} q(i|y)f(y)dy}{\int_{\theta_L}^{\theta^c(i)} q(i|y)f(y)dy} = 1-\theta^c(i), \tag{43}$$

where $q(i|\theta)$ denotes the probability of investment i given type θ (or the density at that point). Since $q(i|\theta) = q(i|\bar{\theta})$ for all $\theta \leq \bar{\theta}$ the claim in (43) is equivalent to

$$(1-\theta)\left(F(\bar{\theta})q(i|\bar{\theta}) + \int_{\bar{\theta}}^{\theta} q(i|y)f(y)dy\right) = (1-\theta^c(i))\left(F(\bar{\theta})q(i|\bar{\theta}) + \int_{\bar{\theta}}^{\theta^c(i)} q(i|y)f(y)dy\right). \tag{44}$$

For $i = i^c(\theta_H)$, we have that $q(i|\theta) = \frac{(1-\theta_H)^2 f(\theta_H)}{(1-\theta)^2 f(\theta)}$ and for all other investments i, it is equal to a fraction with the same denominator and a numerator that only depends on i but not on θ. The numerator thus cancels out and the claim in (43) is equivalent to

$$(1-\theta)\left(\frac{F(\bar{\theta})}{(1-\bar{\theta})^2 f(\bar{\theta})} + \int_{\bar{\theta}}^{\theta} \frac{1}{(1-y)^2}dy\right) = (1-\theta^c(i))\left(\frac{F(\bar{\theta})}{(1-\bar{\theta})^2 f(\bar{\theta})} + \int_{\bar{\theta}}^{\theta^c(i)} \frac{1}{(1-y)^2}dy\right). \tag{45}$$

Since $\bar{\theta}$ maximizes R it holds that $(1-\bar{\theta})f(\bar{\theta}) = F(\bar{\theta})$ (see equation 35). Consequently, the claim in (43) is equivalent to

$$(1-\theta)\left(\frac{1}{1-\bar{\theta}} + \frac{1}{1-\theta} - \frac{1}{1-\bar{\theta}}\right) = (1-\theta^c(i))\left(\frac{1}{1-\bar{\theta}} + \frac{1}{1-\theta^c(i)} - \frac{1}{1-\bar{\theta}}\right), \tag{46}$$

which is true.

□

Proof of Proposition 2.

For any $x_2 : [\theta_L, \theta_H] \to [0,1]$ that is part of an incentive compatible contract, let $\theta^0 \in \Theta$ be the supremum of all types with $x_2(\theta) = 0$. The IC constraints then imply that $u_S(\theta) = u_S(\theta^0)$ for all types $\theta \leq \theta^0$. In the buyer's optimal contract it will then hold that $x_2(\theta) = 0$ and $t(\theta) = S(i^c(1)) - u^c(\theta^0)$ for all $\theta \leq \theta^0$. We therefore now take such a threshold θ^0 as given. Following standard methods of finding an optimal screening contract we replace the IC constraints by the requirement that x_2 is non-decreasing and

$$u_S(\theta) = v(i^c(1)) \int_{\theta^0}^{\theta} x_2(y)dy + u^c(\theta^0). \tag{47}$$

We then define a set of candidate functions as $X^0 := \{x_2 : [\theta^0, \theta_H] \to [0,1],\ \text{nondecreasing}\}$ and write the problem as

$$\max_{x\in X^0} S(i^c(1)) - u^c(\theta^0) - \int_{\theta^0}^{\theta_H} (R'(\theta)+1)x_2(\theta)v(i^c(1))d\theta \tag{48}$$

$$s.t. \quad \int_{\theta^0}^{\theta} x_2(y) - \frac{v(i^c(y))}{v(i^c(1))}dy \geq 0. \tag{49}$$

Because $R'(\theta) = (1-\theta)f(\theta) - F(\theta) \geq -1$, the probability $x_2(\theta)$ must be as small as possible. This suggests that IR should bind everywhere, which would imply that the optimal x_2 is

$$x_2(\theta) = \frac{v(i^c(\theta))}{v(i^c(1))}, \tag{50}$$

which is indeed increasing. Therefore, once we have shown that the IR constraint is binding everywhere, we have found the solution to the optimization problem. To do this, note first that because the objective function in (48) can also be written as

$$S(i^c(1)) - u_S(\theta_H) - \int_{\theta^0}^{\theta_H} R'(\theta)x_2(\theta)v(i^c(1))d\theta \tag{51}$$

and because $R'(\theta) > 0$ for all $\theta < \bar{\theta}$ it follows that $\theta^0 \geq \bar{\theta}$. Furthermore, for the part that depends on x we can use integration by parts to get

$$\begin{aligned} & u_S(\theta_H) + \int_{\theta^0}^{\theta_H} R'(\theta)x_2(\theta)v(i^c(1))d\theta \\ = \;& (1-\theta_H)f(\theta_H)u_S(\theta_H) - R'(\theta^0)u^c(\theta^0) - \int_{\theta^0}^{\theta_H} R''(\theta)u_S(\theta)d\theta \\ \geq \;& (1-\theta_H)f(\theta_H)u^c(\theta_H) - R'(\theta^0)u^c(\theta^0) - \int_{\theta^0}^{\theta_H} R''(\theta)u^c(\theta)d\theta \\ = \;& u^c(\theta_H) + \int_{\theta^0}^{\theta_H} R'(\theta)v(i^c(\theta))d\theta \end{aligned} \tag{52}$$

This shows that the objective function is maximized at the function x defined in equation (50).

Finally, we find the optimal θ^0: Solving

$$\max_{\theta^0} S(i^c(1)) - u^c(\theta^0) - \int_{\theta^0}^{\theta_H} (R'(\theta)+1)v(i^c(\theta))d\theta \tag{53}$$

yields $\bar{\theta}$ as the optimal cut-off value.

Proof of Lemma 2. When the buyer observes an investment $i \in \mathrm{I}^*$, she updates that the seller must have an outside option in $\Theta^*(i)$. The share she offers will therefore also lie in $\Theta^*(i) \subset \{\theta_1, ..., \theta_H\}$, and it will never be more than the highest possible type would accept, i.e., the offer is not higher than $\theta_m = \max \Theta^*(i)$. The profit received by type θ_m from choosing i is therefore equal to $\theta_m v(i) - c(i)$, which would be strictly smaller than $u^c(\theta_m)$ if $i \neq i_m$. Therefore $i = i_m$, which is the same as $\theta^c(i) = \theta_m$. □

Proof of Lemma 3. Let $i_k \in \mathrm{I}^*$. From Lemma 2 we know already that $U(P, i_k, \theta_k) = u^*(\theta_k)$. First, we show that the equality also holds for the lowest type, i.e. that $U(P, i_k, \theta_1) = u^*(\theta_1)$. To this end, let θ_l be the lowest type with this property, i.e., $U(P, i_k, \theta_l) = u^*(\theta_l)$ and $U(P, i_k, \theta) < u^*(\theta)$ for all $\theta < \theta_l$. Since no type below θ_l chooses i_k, the offer following it cannot be lower than θ_l. This implies that every lower type would get the same payoff as type θ_l when investing i_k :

$$U(P, i_k, \theta_l) = v(i_k) \int o dP_{i_k}(o) - c(i_k) = U(P, i_k, \theta) \text{ for all } \theta \leq \theta_l. \tag{54}$$

Payoff monotonicity then implies that $U(P, i_k, \theta) = u^*(\theta)$ for any type $\theta \leq \theta_l$, hence $l = 1$.

Second, we show that for a seller of type θ_{l+1}, $l \geq 1$, the investments that are best responses to P can be found by maximizing $P_i(\theta_l)v(i)$ over all $i \in \mathrm{I}^*$. More precisely, we claim that for all $l \geq 1$

$$\arg\max_{i \in \mathrm{I}^*} U(P, i, \theta_{l+1}) = \arg\max_{i \in \mathrm{I}^*} P_i(\theta_l)v(i) \subset \arg\max_{i \in \mathrm{I}^*} U(P, i, \theta_l). \tag{55}$$

If this claim is true it verifies the lemma, since it implies that

$$i_k \in \arg\max_{i \in \mathrm{I}^*} U(P, i, \theta_k) \subset ... \subset \arg\max_{i \in \mathrm{I}^*} U(P, i, \theta_1). \tag{56}$$

It remains to prove the claim, which we will do by induction. For $l = 1$ we have that

$$\arg\max_{i \in \mathrm{I}^*} U(P, i, \theta_1) = \mathrm{I}^* \tag{57}$$

and for all $i \in \mathrm{I}^*$

$$U(P, i, \theta_2) = u^*(\theta_1) + (\theta_2 - \theta_1)P_i(\theta_1)v(i). \tag{58}$$

This expression is maximized over i whenever $P_i(\theta_1)v(i)$ is maximized.

Assume now that the claim is true for $l \geq 1$. We will show that it also holds for $l+1$. Consider first investments in the set $\mathrm{I}' = \arg\max_{i \in \mathrm{I}^*} U(P, i, \theta_{l+1})$. For all $i \in \mathrm{I}'$ it holds that type θ_{l+2}'s payoff is

$$U(P, i, \theta_{l+2}) = u^*(\theta_{l+1}) + (\theta_{l+2} - \theta_{l+1}) P_i(\theta_{l+1}) v(i), \tag{59}$$

and hence

$$\arg\max_{i \in \mathrm{I}'} U(P, i, \theta_{l+2}) = \arg\max_{i \in \mathrm{I}'} P_i(\theta_{l+1}) v(i). \tag{60}$$

In case that $\mathrm{I}' = \mathrm{I}^*$, we are done, so assume now that the set $\mathrm{I}^* \setminus \mathrm{I}'$ is nonempty. Investments $i \in \mathrm{I}^* \setminus \mathrm{I}'$ are not chosen by type θ_{l+1}, which means that the buyer will not make the offer θ_{l+1} and thus $P_i(\theta_{l+1}) v(i) = P_i(\theta_l) v(i)$. Using the induction hypothesis, we have that for any $i \in \mathrm{I}^* \setminus \mathrm{I}'$ and $i' \in \mathrm{I}'$

$$P_i(\theta_{l+1}) v(i) = P_i(\theta_l) v(i) < P_{i'}(\theta_l) v(i') \leq P_{i'}(\theta_{l+1}) v(i'). \tag{61}$$

This means that investment levels in $\mathrm{I}^* \setminus \mathrm{I}'$ do not maximize $P_i(\theta_{l+1}) v(i)$ and therefore

$$\arg\max_{i \in \mathrm{I}^*} P_i(\theta_{l+1}) v(i) = \arg\max_{i \in \mathrm{I}'} P_i(\theta_{l+1}) v(i). \tag{62}$$

For all $i \in \mathrm{I}^* \setminus \mathrm{I}'$ it holds for type θ_{l+2}'s payoff that

$$U(P, i, \theta_{l+2}) < u^*(\theta_{l+1}) + (\theta_{l+2} - \theta_{l+1}) P_i(\theta_{l+1}) v(i). \tag{63}$$

Comparing this payoff to the payoff from investing $i' \in \mathrm{I}'$ (equation 59), we can conclude that

$$\arg\max_{i \in \mathrm{I}^*} U(P, i, \theta_{l+2}) = \arg\max_{i \in \mathrm{I}'} U(P, i, \theta_{l+2}) \subset \mathrm{I}'. \tag{64}$$

The claim now follows from (60), (64) and (62).

□

Proof of Lemma 4. The first claim follows from Lemma 3, which says that for all $i_m \in \mathrm{I}^*$ it holds that $U(P, i_m, \theta) = u^*(\theta)$ for all $\theta \leq \theta_m$, and hence for all $k < m$

$$u^*(\theta_{k+1}) = u^*(\theta_k) + (\theta_{k+1} - \theta_k) P_{i_m}(\theta_k) v(i_m). \tag{65}$$

To show the second claim of the lemma, note first that for any type θ_k with $i_k \in \mathrm{I}^*$ it must be true that $p_{kk} > 0$, because else $U(P, i_k, \theta_{k-1})$ would be too low: if $p_{kk} = 0$, this payoff would be equal to

$$U(P, i_k, \theta_{k-1}) = \big((1 - p_{kk})\theta_{k-1} + p_{kk}\theta_k\big) v(i_k) - c(i_k) = \theta_{k-1} v(i_k) - c(i_k) < u^c(\theta_{k-1}). \tag{66}$$

Next, assume that for $m > k$ as in the lemma we have $p_{mk} = 0$. Then

$$0 = P_{i_m}(\theta_k)v(i_m) - P_{i_m}(\theta_{k-1})v(i_m) = \frac{u^*(\theta_{k+1}) - u^c(\theta_k)}{\theta_{k+1} - \theta_k} - \frac{u^c(\theta_k) - u^*(\theta_{k-1})}{\theta_k - \theta_{k-1}}, \tag{67}$$

and hence

$$u^c(\theta_k) = u^*(\theta_{k+1})\frac{\theta_k - \theta_{k-1}}{\theta_{k+1} - \theta_{k-1}} + u^*(\theta_{k-1})\frac{\theta_{k+1} - \theta_k}{\theta_{k+1} - \theta_{k-1}}. \tag{68}$$

Since the function u^c is strictly convex and

$$\theta_k = \theta_{k+1}\frac{\theta_k - \theta_{k-1}}{\theta_{k+1} - \theta_{k-1}} + \theta_{k-1}\frac{\theta_{k+1} - \theta_k}{\theta_{k+1} - \theta_{k-1}}, \tag{69}$$

it must on the other hand be true that

$$u^c(\theta_k) < u^c(\theta_{k+1})\frac{\theta_k - \theta_{k-1}}{\theta_{k+1} - \theta_{k-1}} + u^c(\theta_{k-1})\frac{\theta_{k+1} - \theta_k}{\theta_{k+1} - \theta_{k-1}}. \tag{70}$$

Hence, we have found a contradiction to $p_{mk} = 0$ and can conclude that $p_{mk} > 0$. □

Proof of Proposition 3. Let $i_k \in \mathrm{I}^*$. When observing i_k, the buyer's expected profit from offering θ_j is

$$(1 - \theta_j)\frac{\sum_{l=1}^{j} f_l q_{lk}}{\sum_{l=1}^{k} f_l q_{lk}}. \tag{71}$$

We know from Lemma 4 that to be consistent with the seller's behavior, the buyer has to offer all θ_j with $i_j \in \mathrm{I}^*, j \leq k$ with positive probability. Offering θ_k is consistent with profit maximization if

$$\sum_{l=1}^{k} f_l q_{lk}(1 - \theta_k) \geq \sum_{l=1}^{j} f_l q_{lk}(1 - \theta_j) \text{ for all } j, \tag{72}$$

and offering θ_j with positive probability is possible if this condition holds with equality. As a first step, we collect all inequalities that define the buyer's behavior in an equilibrium (P, Q). We let

$$K := \{k : \; i_k \in \mathrm{I}^* \backslash \{i_H\}\} \tag{73}$$

index all chosen investments that are strictly smaller than i_H. We treat H separately because we have to account for the fact that Q is a stochastic matrix, i.e., that the row entries add up to one. From (72) we get that the following condition must hold for all j, k with $j \leq k$ and $k \in K$:

$$\sum_{l=1}^{j} f_l q_{lk}(\theta_k - \theta_j) + \sum_{l=j+1}^{k} f_l q_{lk}(\theta_k - 1) \leq 0, \tag{74}$$

with equality if $j \in K$. Moreover, we have for all $j \leq k \in K$

$$q_{jk} \geq 0. \tag{75}$$

In addition, we have the same condition for the highest investment i_H:

$$\sum_{l=1}^{j} f_l q_{lH}(\theta_H - \theta_j) + \sum_{l=1}^{H} f_l q_{lH}(\theta_H - 1) \leq 0 \tag{76}$$

for all $j < H$, with equality if $j \in K$. We also again have $q_{jH} \geq 0$ for all $j \leq H$. Plugging in $q_{lH} = 1 - \sum_{l \leq k \in K} q_{lk}$ we get that for all $j < H$

$$\sum_{l=1}^{j} \sum_{l \leq k \in K} q_{lk} f_l(\theta_j - \theta_H) + \sum_{l=j+1}^{H-1} \sum_{l \leq k \in K} q_{lk} f_l(1 - \theta_H) \leq R(\theta_H) - R(\theta_j), \tag{77}$$

with equality if $j \in K$, as well as

$$\sum_{j \leq k \in K} q_{jk} \leq 1 \tag{78}$$

We are going to treat the variables q_{jk} as one big vector, denoted by q. The entries in q are indexed by $jk, k \in K, 1 \leq j \leq k$, hence $\mathrm{q} \in \mathbb{R}^n$ with $n = \sum_{k \in K} k$. Similarly, we define a vector $\mu^{jk} \in \mathbb{R}^n$ by $\mu^{jk}_{lk} = f_l(\theta_k - \theta_j)$ for all $l \leq j$ and $\mu^{jk}_{lk} = f_l(\theta_k - 1)$ for all $l > j$ and zero else. Furthermore, we define a vector μ^j by $\mu^j_{lk} = f_l(\theta_j - \theta_H)$ for all $l \leq j$ and $\mu^j_{lk} = f_l(1 - \theta_H)$ for all $l > j$. Last, let 1^j denote a vector with $1^j_{jk} = 1$ for $j \leq k \in K$ and 0 else; and let e^{jk} be a vector with $e^{jk}_{jk} = 1$ and 0 else.
With these definitions, our inequalities (75), (78), (74), (77) read

$$-e^{jk}\mathrm{q} \;\leq\; 0 \qquad 1 \leq j \leq k, k \in K \tag{79}$$

$$1^j\mathrm{q} \;\leq\; 1 \qquad j = 1, ..., H-1 \tag{80}$$

$$\mu^{jk}\mathrm{q} \;\leq\; 0 \quad \text{for all } k \in K, j < k \text{ and also} \geq 0 \text{ for } j \in K \tag{81}$$

$$\mu^{j}\mathrm{q} \;\leq\; R(\theta_H) - R(\theta_j) \quad j < H \text{ and also } \geq 0 \text{ for } j \in K. \tag{82}$$

If the entries q_{jk} of the vector q are part of an equilibrium, then q constitutes a solution of this system of inequalities. As the next step in the proof, we find a system of inequalities that is an alternative of this system, i.e, we find a system that has no solution if and only if this one has a solution. We use Theorem 22.1 in Rockafellar (1970) to get the following alternative system:

$$\sum_{j=1}^{H-1} \beta_j + \sum_{l=1}^{H-1} \delta_l (R(\theta_H) - R(\theta_l)) \quad < \quad 0 \tag{83}$$

$$\sum_{j=1}^{H-1} 1^j \beta_j + \sum_{jk} \mu^{jk} \gamma_{jk} + \sum_{l=1}^{H-1} \mu^l \delta_l \quad \geq \quad 0, \tag{84}$$

where we are looking for coefficients $\beta_j \geq 0$, $j = 1, ..H-1$, γ_{jk} (≥ 0 if $j \notin K$), and δ_l (≥ 0 if $l \notin K$). That is, if there exists an equilibrium of the signaling game, then there are no such β, γ, δ that satsify inequalities (83) and (84). Note that (84) concerns the components of a vector of dimension n. For the subsequent analysis, it is convenient to write this inequality as an inequality in each coefficient jk with $k \in K$ and $j \leq k$

$$\beta_j + \sum_{i=1}^{j-1} \gamma_{ik} f_j (\theta_k - 1) + \sum_{i=j}^{k-1} \gamma_{ik} f_j (\theta_k - \theta_i) + \sum_{l=1}^{j-1} \delta_l f_j (1 - \theta_H) + \sum_{l=j}^{H-1} \delta_l f_j (\theta_l - \theta_H) \geq 0 \tag{85}$$

Let $\hat{k} = \min K$. We claim that $\bar{k} = \hat{k}$ and first show that $R(\theta_l) \leq R(\theta_{\hat{k}})$ for all $l < \hat{k}$. Assume not. Then we can construct a solution by setting $\delta_l = \gamma_{lk} = 1$ and $\delta_{\hat{k}} = \gamma_{\hat{k}k} = -1$ for all $k \in K$ and all other coefficients equal to zero. Plugging in these values, the first inequality (83) reads $(R(\theta_H) - R(\theta_l)) - (R(\theta_H) - R(\theta_{\hat{k}})) < 0$ and is satisfied because we assumed that $R(\theta_l) > R(\theta_{\hat{k}})$. The value of the second inequality (85) depends on how $\hat{k}$ and l compare to j and k, where due to the definition of $\hat{k}$ it holds that $k \geq \hat{k} > l$. This second inequality is equal to $\theta_k - 1 - \theta_k + 1 + 1 - \theta_H - 1 + \theta_H \geq 0$ if $j > \hat{k}$, equal to $\theta_k - 1 - \theta_k + \theta_{\hat{k}} + 1 - \theta_H + \theta_H - \theta_{\hat{k}} \geq 0$ if $l < j \leq \hat{k}$, and $\theta_k - \theta_l - \theta_k + \theta_{\hat{k}} - \theta_H + \theta_l + \theta_H - \theta_{\hat{k}} \geq 0$ if $j \leq l$. All of these simplify to $0 \geq 0$.

Similarly, one can show that $R(\theta_{\hat{k}+1}) \leq R(\theta_{\hat{k}})$ is also necessarily true, because else there would be a solution with $\delta_{\hat{k}+1} = \gamma_{\hat{k}+1k} = 1$ and $\delta_{\hat{k}} = \gamma_{\hat{k}k} = -1$. With these values, the first inequality would read $(R(\theta_H) - R(\theta_{\hat{k}+1})) - (R(\theta_H) - R(\theta_{\hat{k}})) < 0$, which holds if $R(\theta_{\hat{k}+1}) > R(\theta_{\hat{k}})$. As above, the second inequality would be true with equality in all three cases ($j > \hat{k}+1$, $\hat{k} < j \leq \hat{k}+1$, and $j \leq \hat{k}$). Hence, we have shown that $\hat{k} = \bar{k}$. [15]

[15] Note that we could have shown more generally that $K \subset \{k \text{ with } R(\theta_k) \geq R(\theta_{k+1})\}$.

Next we show that K is an interval of consecutive numbers. Assume to the contrary that there is a gap in K, i.e, that there exist $l < m < h$ with $m \notin K$, $l = \max\{k \in K, k \leq m\}$ and $h = \min\{k \in K, k \geq m\}$. There is a $\lambda \in (0,1)$ with $(1-\lambda)\theta_h + \lambda\theta_l = \theta_m$. Define $\delta_l = \gamma_{lk} = -\lambda, \delta_m = \gamma_{mk} = 1, \delta_h = \gamma_{hk} = -(1-\lambda)$ for all relevant $k \in K$. Then the first condition holds because R is concave on K: $\lambda R(\theta_l) + (1-\lambda)R(\theta_h) - R(\theta_m) < 0$. That the second condition always holds with equality is seen immediately if $k \leq l$, for which this condition takes the form $\theta_m - \theta_H - \lambda(\theta_h - \theta_H) - (1-\lambda)(\theta_l - \theta_H) = 0$. For the remaining case $k \geq h$ there has to be again a case distinction regarding j, each case leading to the same result $0 \geq 0$. Thus concavity of R implies that there are no gaps in chosen investment, $K = \{\bar{k}, ..., H-1\}$.

□

Proof of Proposition 4.

The buyer's strategy was constructed to make the seller indifferent between the investments in the support of his strategy. In this proof, we are therefore concerned with the seller's strategy making the buyer indifferent between the offers in the support of her strategy. Following investment i_k, the buyer is indifferent between all offers in $\{\theta_{\bar{k}}, ..., \theta_k\}$ if

$$(1-\theta_l)\sum_{j=1}^{l} f_j q_{jk} = (1-\theta_{\bar{k}})\sum_{j=1}^{\bar{k}} f_j q_{jk} \quad \text{for all } k \geq l > \bar{k}. \tag{86}$$

With our definition of the seller's strategies, we have for the right-hand side

$$(1-\theta_{\bar{k}})\sum_{j=1}^{\bar{k}} f_j q_{jk} = \lambda_k - \lambda_{k+1} \tag{87}$$

and for the left-hand side

$$\begin{aligned}(1-\theta_l)\sum_{j=1}^{l} f_j q_{jk} &= (1-\theta_l)\left(\sum_{j=1}^{\bar{k}} f_j \frac{\lambda_k - \lambda_{k+1}}{R(\theta_{\bar{k}})} + \sum_{j=\bar{k}+1}^{l} f_j \frac{\lambda_k - \lambda_{k+1}}{\lambda_j}\right) \\ &= (1-\theta_l)(\lambda_k - \lambda_{k+1})\left(\frac{1}{1-\theta_{\bar{k}}} + \sum_{j=\bar{k}+1}^{l}\left(\frac{1}{1-\theta_j} - \frac{1}{1-\theta_{j-1}}\right)\right) \\ &= \lambda_k - \lambda_{k+1}.\end{aligned} \tag{88}$$

The buyer prefers $\theta_{\bar{k}}$ to $\theta_l < \theta_{\bar{k}}$ if

$$(1-\theta_l)\sum_{j=1}^{l} f_j q_{jk} \leq (1-\theta_{\bar{k}})\sum_{j=1}^{\bar{k}} f_j q_{jk}. \tag{89}$$

With $q_{jk} = q_{\bar{k}k}$ this condition reads

$$(1-\theta_l)F(\theta_l) \leq (1-\theta_{\bar{k}})F(\theta_{\bar{k}}) \quad \text{for all } l < \bar{k} \tag{90}$$

and hence is satisfied.

In the remainder of this proof we show that all $q_{jk} \geq 0$ and that they add up to one.

First, note that

$$\frac{R(\theta_k) - R(\theta_{k-1})}{\theta_k - \theta_{k-1}} = \frac{f_k(1-\theta_{k-1})}{(\theta_k - \theta_{k-1})} - F(\theta_k) = \frac{f_k(1-\theta_k)}{(\theta_k - \theta_{k-1})} - F(\theta_{k-1}) \tag{91}$$

and therefore

$$\lambda_k - \lambda_{k+1} = (1-\theta_k)\left(\frac{R(\theta_k) - R(\theta_{k-1})}{\theta_k - \theta_{k-1}} - \frac{R(\theta_{k+1}) - R(\theta_k)}{\theta_{k+1} - \theta_k}\right) \geq 0, \tag{92}$$

where the latter holds because R is concave on $\{\theta_{\bar{k}}, ..., \theta_H\}$. Note further that for $k = \bar{k}$

$$R(\theta_{\bar{k}}) \geq \lambda_{\bar{k}+1} \Leftrightarrow (\theta_{\bar{k}+1} - \theta_{\bar{k}})F(\theta_{\bar{k}}) \geq f_{\bar{k}+1}(1-\theta_{\bar{k}+1}) \Leftrightarrow R(\theta_{\bar{k}}) \geq R(\theta_{\bar{k}+1}). \tag{93}$$

Last,

$$\sum_{k=j}^{H} q_{jk} = \sum_{k=j}^{H} \frac{\lambda_k - \lambda_{k+1}}{\lambda_j} = 1 \qquad \text{for all } j > \bar{k} \tag{94}$$

$$\sum_{k=\bar{k}}^{H} q_{j\bar{k}} = 1 - \frac{\lambda_{\bar{k}+1}}{R(\theta_{\bar{k}})} + \sum_{k=\bar{k}+1}^{H} \frac{\lambda_k - \lambda_{k+1}}{R(\theta_{\bar{k}})} = 1 \text{ for all } j \leq \bar{k}. \tag{95}$$

□

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